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COMPOSITION TOPOLOGICAL ISOMORPHISM OPERATORS ON SPACES l_2^s

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Abstract: Many results of sequence space l_2^s, $s \in \mathbb{R}$ that are quasi-Hilbert spaces, are studied. Composition of bounded linear operators on these spaces is proved as Fredholem topological isomorphism operators.

Key words and phrases: Quasi-Hilbert space, Fredholem operator, Topological isomorphism,Composition operator.

КОМПОЗИЦИЯ ТОПОЛОГИЧЕСКИЙ ИЗОМОРФИЗМ ОПЕРАТОРОВ В ПРОСТРАНСТВАХ l_2^s

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Аннотация: В статье, изучены многие результаты пространства последовательностей l_2^s, $s \in \mathbb{R}$, которые являются квазигильбертовыми пространствами. Доказано, что композиция ограниченных линейных операторов в этих пространствах есть топологический фредгольмский изоморфизм.

Ключевые слова: Квазигильбертово пространство, фредгольмский оператор, топологический изоморфизм, композиция оператора.

1. INTRODUCTION

It is known, sequence spaces l_p, $0 < p \leqslant \infty$ are Banach space only when $1 < p \leqslant \infty$ and the space l_2 is only one that be Hilbert space[1,2]. In [3,4], we were introduced spaces l_p to a power of a real number, $0 < p < \infty$ which are defined:

$$l_p^s =: \left\{ u = \{u_k\} : \sum_{k=1}^{\infty} \left(\lambda_k^{\frac{s}{2}} |u_k| \right)^p < +\infty \right\},$$

where $\{\lambda_k\} \subset \mathbb{R}_+$, $k \in \mathbb{N}$ such that $\lim_{k\to\infty} \lambda_k = +\infty$ is monotonically increasing sequence. When $s = 0$ then $l_p^0 = l_p$. Also, we were introduced bounded linear operators $Tv = \lambda_k v_k$ and its inverse $T^{-1}w = \lambda_k^{-1} w_k$.

In [5], using a notion of quasi-Banach space, a concept of quasi-Hilbert space which is generalization of Hilbert space were presented. Not all Spaces l_p^s that be quasi-Hilbert spaces .

In this work, we study some of the properties of l_2^s and define operators with their inverse on these spaces. Composition of operators are introduced with some examples, remarks and results, are proved as Fredholem topological isomorphism.

2. SOME RESULTS ABOUT SEQUENCE SPACES l_2^s

A linear space $\mathfrak{U}$ and function ${}_q\|\cdot\|$, that differs from function $\|\cdot\|$ by an inequality: ${}_q\|\xi+\eta\| \leqslant \rho({}_q\|\xi\| +_q \|\eta\|)$, $\forall\xi,\eta \in \mathfrak{U}$, $\rho \in [1,+\infty)$ is called a quasi-normed space.If $\rho = 1$, a quasi-norm ${}_q\|\cdot\|$ be a norm $\|\cdot\|$. A complete quasi-normed space is called a quasi-Banach space [6].

Definition 2.1 [5]. A quasi-Hilbert space $\mathfrak{U}$ is quasi-Banach space such that:

$$ {}_q\|\xi+\eta\|^4 -_q \|\xi-\eta\|^4 = 8({}_q\|\xi\|^2\tau(\xi,\eta) +_q \|\eta\|^2\tau(\eta,\xi)), \forall\xi,\eta \in \mathfrak{U}, \tag{1}$$

is satisfied, where $\tau(\xi,\eta)$ and $\tau(\eta,\xi)$ are Gateaux derivatives:

$$\tau(\xi,\eta) = \frac{{}_q\|\xi\|}{2}\left(\lim_{\varphi\to+0}\frac{{}_q\|\xi+\varphi\eta\| -_q \|\xi\|}{\varphi} + \lim_{\varphi\to-0}\frac{{}_q\|\xi+\varphi\eta\| -_q \|\xi\|}{\varphi}\right), \tag{2}$$

φ is a real number . $\tau(\eta,\xi)$ at $\eta \in \mathfrak{U}$ in the direction ξ is defined similarly. If

$$\tau(\xi,\eta) =_q \|\xi\| \left(\lim_{\varphi\to0}\frac{{}_q\|\xi+\varphi\eta\| -_q \|\xi\|}{\varphi}\right), \tag{3}$$

then $\mathfrak{U}$ is a smooth quasi-Hilbert space.

Remark 2.2. A quasi-Banach space $\mathfrak{U}$ is a Hilbert space if and only if an equality:

$$ {}_q\|\xi+\eta\|^2 +_q \|\xi-\eta\|^2 = 2_q\|\xi|^2 + 2_q\|\eta\|^2, \forall\xi,\eta \in \mathfrak{U}, \tag{4}$$

is satisfied. Also, a Hilbert space is a quasi-Hilbert space [5], conversely, is not true, indeed:

Suppose $\xi,\eta \in l_4^1$, where $\xi = \{\xi_k\} = \{1,1,0,...\}$, $\eta = \{\eta_k\} = \{1,1,0,...\}$ with $\{\lambda_k\} = \{\sqrt{k}\}$. Then, the space l_4^1 is a smooth quasi-Hilbert space, since equations (1) and (3) are hold, but an equation (4) is not satisfied.

Theorem 2.3 [5]. l_2^s is a smooth quasi-Hilbert space and Hilbert space with a function: ${}_q\|\xi\|_2^s = \left(\sum\limits_{k=1}^{\infty}\lambda_k^s|\xi_k|^2\right)^{1/2}$, $s \in \mathbb{R}$,

Remarks 2.4. (1) If it is replaced s by $s-2$ or $s+2$, in definition of l_2^s we have sequence spaces l_2^{s-2} and l_2^{s+2}. By analogy with Theorem 2.3, these spaces be Hilbert spaces.

(2) Since $d(\xi,\eta) = {}_q\|\xi-\eta\|_2^s = \left(\sum\limits_{k=1}^{\infty}\lambda_k^s|\xi_k-\eta_k|^2\right)^{1/2}$, l_2^s is a complete metric space.

Theorem 2.5. For every $s \in \mathbb{R}$, then $l_2^s \subseteq l_2^{s-2}$ and ${}_q\|v\|_2^{s-2} \leqslant C.{}_q\|v\|_2^s$ where $C > 0$.

Proof: Let $v = \{v_k\} \in l_2^s$, so $\sum\limits_k \lambda_k^s|v_k|^2 < +\infty$. Since $\lambda_k^{s-2}|v_k|^2 \leqslant \lambda_k^s|v_k|^2$,where $\lambda_k > 0$, then $\sum\limits_k \lambda_k^{s-2}|v_k|^2 \leqslant \sum\limits_k \lambda_k^s|v_k|^2 < \infty$ $\forall k \in \mathbb{N}$, that is, $v \in l_2^{s-2}$, which implies that $l_2^s \subset l_2^{s-2}$.

Now, $\forall k \in \mathbb{N}$, we have ${}_q\|v\|_2^{s-2} = (\sum\limits_k \lambda_k^{s-2}|v_k|^2)^{\frac{1}{2}}$

$$= (\sum_k \lambda_k^{-2}\lambda_k^s|v_k|^2)^{\frac{1}{2}} \leqslant (\sup_k \lambda_k^{-2})^{\frac{1}{2}}.{}_q\|v\|_2^s.$$

So ${}_q\|v\|_r^{s-2} \leqslant (\sup\limits_k \lambda_k^{-2})^{\frac{1}{2}}.{}_q\|v\|_2^s$. Putting $C = (\sup\limits_k \lambda_k^{-2})^{\frac{1}{2}}$, we have the desired result.

Theorem 2.6. For every $s \in \mathbb{R}$, then $l_2^{s+2} \subseteq l_2^s$ and ${}_q\|v\|_2^s \leqslant C.{}_q\|v\|_2^{s+2}$ where $C > 0$.

Proof: Let $v = \{v_k\} \in l_2^{s+2}$,then $\sum\limits_k \lambda_k^{s+2}|v_k|^2 < +\infty$, $\forall k \in \mathbb{N}$. Since, ${}_q\|v\|_2^s = (\sum\limits_k \lambda_k^s|v_k|^2)^{\frac{1}{2}} = (\sum\limits_k \lambda_k^{s+2-2}|v_k|^2)^{\frac{1}{2}} = (\sum\limits_k \lambda_k^{s+2}\lambda_k^{-2}|v_k|^2)^{\frac{1}{2}} \leqslant (\sup\limits_k \lambda_k^{-2})^{\frac{1}{2}}.{}_q\|v\|_2^{s+2}$, this implies that ${}_q\|v\|_2^s \leqslant C.{}_q\|v\|_2^{s+2}$, where $C = (\sup\limits_k \lambda_k^{-2})^{\frac{1}{2}}$ and the proof is compete.

COMPOSITION LINEAR OPERATORS

If $\mathfrak{U}$ and $\mathfrak{V}$ are normed spaces, a linear operator $T : \mathfrak{U} \to \mathfrak{V}$ is bounded if $\|T\xi\|_{\mathfrak{V}} \leqslant \gamma.\|\xi\|_{\mathfrak{U}}$ such that a constant $\gamma \geqslant 0$;linear isometric if $\|T\xi\|_{\mathfrak{V}} = \|\xi\|_{\mathfrak{U}}$. A set of all bounded linear operators is denoted by $\mathbf{B}(\mathfrak{U}, \mathfrak{V})$. A bounded operator T whose inverse is bounded, is called topological isomorphism. If $T \in \mathbf{B}(\mathfrak{U}, \mathfrak{V})$ and $S \in \mathbf{B}(\mathfrak{V}, \mathfrak{W})$, then a composition of these operators is $C_T(S) = SoT : \mathfrak{U} \to \mathfrak{W}$, where $(SoT)\xi = S(T\xi)$, $\forall \xi \in \mathfrak{U}$ [1,3] .

Remark 3.1. If T is a linear isometric operator then $\|T0\|_{\mathfrak{V}} = \|0\|_{\mathfrak{U}}$, which implies that (kernel of T)$kerT = \{0\}$, and T is injective. Also, since $\gamma = \|T\| = 1$, then T is bounded.

Example 3.2. If an operator $T : l_2^0 \to l_2^0$ is given by: $T\xi = \xi$, $\forall \xi \in l_2^0$ then it is topological isomorphism, since it is bounded bijective and $\|T\xi\|_2^0 = \|\xi\|_2^0$.

Theorem 3.3. For every $s \in \mathbb{R}$, A linear operator $T_1 : l_2^s \to l_2^{s-2}$ such that $T_1 u = \lambda_k u_k$ is topological isomorphism.

Proof: Let $u = \{u_k\} \in l_2^s$, $\|T_1 u\|_2^{s-2} = (\sum\limits_k \lambda_k^{s-2} |\lambda_k u_k|^2)^{\frac{1}{2}}$

$$= (\sum_k \lambda_k^{s-2}.\lambda_k^2 |u_k|^2)^{\frac{1}{2}} = (\sum_k \lambda_k^s |u_k|^2)^{\frac{1}{2}} = \|u\|_2^s,$$

where $\|T_1\| = 1$, then T_1 is bounded. Also, $\forall v = \{v_k\} \in l_2^{s-2}$,

$\|T_1^{-1} v\|_2^s = (\sum\limits_k \lambda_k^s |\lambda_k^{-1} v_k|^2)^{\frac{1}{2}} = (\sum\limits_k \lambda_k^{s-2}.|u_k|^2)^{\frac{1}{2}} = \|v\|_2^{s-2}$, where $\|T_1^{-1}\| = 1$.Then T_1^{-1} is bounded, hence T_1 is topological isomorphism.

Theorem 3.4.A linear operator $T_2 : l_2^{s+2} \to l_2^s$, $T_2 u = \lambda_k u_k$, $k \in \mathbb{N}$ is topological isomorphism.

Proof: Proof of this theorem is similar to proof of Theorem 3.3, where $T_2^{-1} v = \lambda_k^{-1} v_k$,$\forall v = \{v_k\} \in l_2^s$is an inverse of bounded operator which is also bounded.

Definition 3.5 [7]. $T \in \mathbf{B}(\mathfrak{U}, \mathfrak{V})$, where $\mathfrak{U}$ and $\mathfrak{V}$ are Banach spaces, is called a Fredholm operator if $kerT$ and $cokerT = \mathfrak{F}/imgT$ are finite, that is, an index of T $(indT) = dim\ kerT - dim\ cokerT$is finite.

Remark 3.6. If $T \in \mathbf{B}(\mathfrak{U}, \mathfrak{V})$ is topological isomorphism, then $ker(T) = \{0\}$ and $im(T) = \mathfrak{F}$, that is, $coker(T) := \mathfrak{F}/im(T) = \{0\}$, hence $indT = 0$, so T is Fredholem, conversely, is not true, indeed:

let $T : l_2^0 \to l_2^0$ be an operator such that $T(\xi_1, \xi_2, \xi_3, ...) = (0, \xi_2, \xi_3, ...)$.

Clearly, $kerT = \{0\}$, so $dimkerT = 0$. Also, $imgT = Span\{\xi_2, \xi_3, ...\}$

$\neq l_2^0$, then T is not surjective and $dim\ cokerT = 1$, hence

$ind\ T = 0$, so T is Fredholem and has no inverse, then T is not topological isomorphism.

Theorem 3.7. For every $s \in \mathbb{R}$, a linear operator $T_1 : l_2^s \to l_2^{s-2}$, $T_1 = \lambda_k u_k$,$k \in \mathbb{N}$ is a Fredholem operator.

Proof: From Theorem 3.4, T_1 is bijective operator. Since (image of T_1) $img\ T_1 = l_2^{s-2}$ where T_1 is surjective, then $coker\ T_1 = l_2^{s-2}/img\ T_1 = 0$, so $dim\ coker\ T_1 = 0$. Also, T_1 is injective, this implies that $ker\ T_1 = 0$ and $dim\ kerT_1 = 0$. Thus, $ind\ T_1 = 0$, so T_1 is a Fredholem operator.

Remark 3.8. Similar to Theorem 3.7, we can prove an operator T_2 is Fredholem, where $indT_2 = 0$.

Theorem 3.9 [8]. If $T : \mathfrak{U} \to \mathfrak{V}$ and $S : \mathfrak{V} \to \mathfrak{W}$ are Fredholem operators, then $C_T(S)$ is Fredholem operator such that $ind(C_T(S)) = ind(S) + ind(T)$.

Theorem 3.10. A composition operator $C_{T_2}(T_1) : l_2^{s+2} \to l_2^{s-2}$,$s \in \mathbb{R}$, is topological isomorphism and Fredholem.

Proof: Let $u = \{u_k\} \in l_2^{s+2}$, we have,

$$\|T_1(T_2u_k)\|_2^{s-2} = (\sum_k \lambda_k^{s-2}|\lambda_k(T_2u_k)|^2)^{\frac{1}{2}} = (\sum_k \lambda_k^{s-2}.\lambda_k^2|\lambda_k u_k|^2)^{\frac{1}{2}}$$

$$= (\sum_k \lambda_k^{s+2}|u_k|^2)^{\frac{1}{2}} = \|u\|_2^{s+2},$$

so $C_{T_2}(T_1)$ is bounded. Also, $\forall v = \{v_k\} \in l_2^{s-2}$,

$$\|T_2^{-1}(T_1^{-1}v_k)\|_2^{s+2} = (\sum_k \lambda_k^{s+2}|\lambda_k^{-1}(T_1^{-1}v_k)|^2)^{\frac{1}{2}} = (\sum_k \lambda_k^{s+2}.\lambda_k^{-2}|\lambda_k^{-1}v_k|^2)^{\frac{1}{2}}$$

$$= (\sum_k \lambda_k^{s-2}|v_k|^2)^{\frac{1}{2}} = \|v\|_2^{s-2},$$

so $(C_{T_1}(T_2))^{-1}$ is bounded. Thus, $C_{T_2}(T_1)$ is topological isomorphism.

According to Theorem 3.7, Remark 3.8 and Theorem 3.9 then $C_{T_2}(T_1)$ is Fredholem, where $ind(C_{T_2}(T_1)) = 0$.

Remark 3.11. If we takeT_1,$T_2 \in \mathbf{B}(l_2^s, l_2^s)$ such that $T_1u = T_2u = \lambda_k u_k$, then, obviously, as above results, operators $C_{T_2}(T_1)= T_1oT_2 : l_2^s \to l_2^s$ and $C_{T_1}(T_2)= T_2oT_1 : l_2^s \to l_2^s$ are topological isomorphism and Fredholem operators.

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