UDK 519.146 + 514.11

ON PARTICULAR DIAMETER BOUNDS FOR INTEGRAL POINT SETS IN HIGHER DIMENSIONS*

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Поступила в редакцию 15.12.2024 г.

Abstract: An integral point set $\mathcal{P}$ is a set of n points in the m-dimensional Euclidean space $\mathbb{R}^m$ with pairwise integral distances, such that $\mathcal{P}$ is not contained in a $(m-1)$-dimensional hyperplane. In the present paper we discuss some classes of planar integral point sets ($m = 2$); we mostly focus on the subsets of a union of two straight lines: facher, rails, scissors and pyramid sets. We use rails and pyramid sets to construct bounds for a diameter of integral point sets in higher dimensions.

Key words and phrases: integral point set, discrete combinatorial geometry, diameter.

О НЕКОТОРЫХ ОЦЕНКАХ НА ДИАМЕТР ЦЕЛОУДАЛЁННЫХ МНОЖЕСТВ В МНОГОМЕРНЫХ ПРОСТРАНСТВАХ

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Аннотация: Целоудалённое множество $\mathcal{P}$ есть такое множество из n точек в m-мерном евклидовом пространстве $\mathbb{R}^m$, что все попарные расстояния между точками есть целые числа и $\mathcal{P}$ не содержится ни в какой $(m-1)$-мерной гиперплоскости. В данной статье обсуждаются некоторые классы плоских целоудалённых множеств ($m = 2$); преимущественно рассматриваются подмножества объединений двух прямых: веерные, рельсовые, ножницеподобные и пирамидальные множества. Рельсовые и пирамидальные множества используются для оценок минимального диаметра целоудалённых множеств в многомерных пространствах.

Ключевые слова: целоудалённое множество, множество с целочисленными расстояниями, дискретная комбинаторная геометрия, диаметр.

1. INTRODUCTION

The study of integral point sets belongs to the classical analysis. It is motivated by both pure theoretical aspects [1], [2] and applications [3].

Definition 1. *A planar integral point set (PIPS) is a set $\mathcal{P}$ of non-collinear points in the plane $\mathbb{R}^2$ such that for any pair of points $P_1, P_2 \in \mathcal{P}$ the Euclidean distance $|P_1P_2|$ between points P_1 and P_2 is integral.*

* This work was carried out at Voronezh State University and supported by Russian Science Foundation grant 19-11-00197.

Definition 1 can be generalized as follows:

Definition 2. *An integral point set $\mathcal{P}$ is a set of n points in the m-dimensional Euclidean space $\mathbb{R}^m$ with pairwise integral distances, such that $\mathcal{P}$ is not contained in a $(m-1)$-dimensional hyperplane.*

How can we characterize an integral point set? First of all, we can look at its dimension; then, we can find its cardinality, which is always finite [4], [5]; finally, we can compute the diameter of a finite point set, which is naturally defined as follows:

Definition 3. *The diameter of an integral point set $\mathcal{P}$ is defined as*

$$\operatorname{diam}(\mathcal{P}) = \max_{P_1, P_2 \in \mathcal{P}} |P_1 P_2|. \tag{1}$$

Every integral point set has also a characteristic [6], [7].

Definition 4. *The characteristic of a PIPS $\mathcal{P}$ is a squarefree number q such that the area of any triangle $M_1 M_2 M_3$, $\{M_1, M_2, M_3\} \subset \mathcal{P}$, is commensurable with $\sqrt{q}$.*

The following notion was introduced in [8].

Definition 5. *The function $d(m,n)$ is the minimal possible diameter of an integral point set $\mathcal{P}$ of n points in m-dimensional Euclidean space $\mathbb{R}^m$.*

The computation of exact values of $d(m,n)$ is a difficult problem. In 2003, Solymosi proved [9] that $d(2,n) \geqslant cn$ for some constant c; now it is known [10] that $c > 5/11$. The most recent advances for higher dimensions can be found in [11]. So, even finding the estimates is not easy.

A long list of known bounds for $d(m,n)$ and some exact values can be found in [8, Theorem 1] or in [12]. Below we will discuss the following ones presented at [8]:

$$d(m, 2m+1) \leqslant 8 \tag{2}$$

$$d(m, 2m+2) \leqslant 13 \tag{3}$$

$$d(m, 3m) \leqslant 109 \tag{4}$$

and the following theorem [8, Theorem 2.1].

Теорема 1. *Let $\mathcal{P}$ be a planar integral point set consisting of $n-2$ points on the line l_1 and two points P_1 and P_2 on a parallel line l_2 with distance r between l_1 and l_2. If there exist positive integers v, w with $f^2 + v^2 = w^2$ and $v < 2r$, where $|P_1 P_2| = f$, then*

$$d(m, n + 2(m-2)) \leqslant \max(w, \operatorname{diam}(\mathcal{P})) \tag{5}$$

In this paper we further study the various features of IPS. In Section 2, we discuss the classification of planar integral points sets; in Sections 3 and 4, we present some upper bounds for $d(m,n)$ based on planar integral point sets of special types and provide some general constructions of such bounds. All the bounds, like (2)–(4), are of the form $d(m, km+p) \leqslant c_{k,p}$.

2. CLASSIFICATION OF PLANAR INTEGRAL POINT SETS

2.1. Integral point sets situated on two straight lines

Definition 6. *A planar integral point sets of n points with $n-1$ points on a straight line is called a facher set.*

Most of the examples of planar integral point sets turns to be facher sets. In [13], facher sets with characteristic 1 are called *semi-crabs*. For $9 \leqslant n \leqslant 122$, the diameter $d(2,n)$ is attained on a facher point set [14]. Facher sets have relatively simple structure and thus they are rather well-investigated [15], [16].

Non-facher integral point sets situated on two straight lines can be further classificated in the following three cases:

Definition 7. *A planar integral point set situated on two parallel straight lines is called a rails set.*

Among the rails sets, sets with 2 points on one line and all the other on another line prevail. In Section 3 below, some examples of such sets are discussed; they include an example of rails sets with 2 points on one line and 38 point on the other line (Fig. 19) and an example of rails set with 4 points on one line and 4 points on the other (Fig. 21). We still do not know whether there are rails set with 4 points on one line and 5 points on the other; and the same for rails sets with 3 points on one line and 9 points on the other.

For the sake of brevity, we will hereafter use the notation $\sqrt{p}/q * \{(x_1,y_1), ..., (x_n, y_n)\}$ from [17], [18], [12], which means that each abscissa is multiplied by $1/q$ and each ordinate is multiplied by $\sqrt{p}/q$, that is

$$\sqrt{p}/q * \{(x_1,y_1), ..., (x_n, y_n)\} = \left\{ \left(\frac{x_1}{q}, \frac{y_1\sqrt{p}}{q} \right), ..., \left(\frac{x_n}{q}, \frac{y_n\sqrt{p}}{q} \right) \right\}.$$

On Fig. 1 we show the following rails set with 3 points on one line and 8 points on the other:

$$\mathcal{P}_{3,8} = \sqrt{255255}/2 * \{(1767; -3); (2791; -3); (4071; -3); \\ (-306; 0); (0; 0); (1798; 0); (2304; 0); (2760; 0); (3534; 0); (4040; 0); (4558; 0)\}$$

It's notable that $\operatorname{char} \mathcal{P}_{3,8} = 255255 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$ is a product of the first 6 odd primes, and the distances between the three points on the lower line are $512 = 2^9$, $640 = 2^7 \cdot 5$ and $512 + 640 = 1152 = 2^7 \cdot 3^2$.

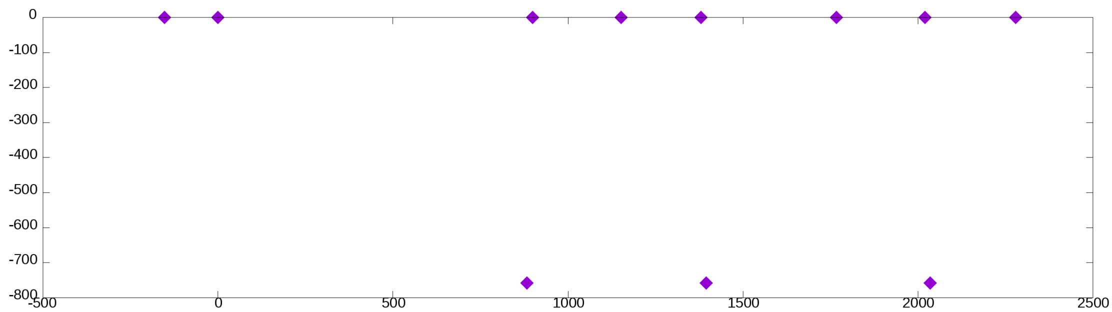

Рис. 1. *PIPS of cardinality 11 and diameter 2423*

Rails sets with 4 and 4 points on the lines are somewhat rather common; an example of such a set is shown on Fig. 2:

$$\sqrt{770}/1 * \{(0; 0); (380; 0); (731; 0); (1111; 0); (354; -12); (377; -12); (734; -12); (757; -12)\}$$

If a rails IPS $\mathcal{P}$ is contained in two parallel lines m_1 amd m_2 and $M_1,M_2 \in \mathcal{P} \cap m_1$, then $|M_1M_2| \geqslant 3$ (the second condition is essential) [19].

Definition 8. *A planar integral point set situated on two perpendicular straight lines is called a cross set.*

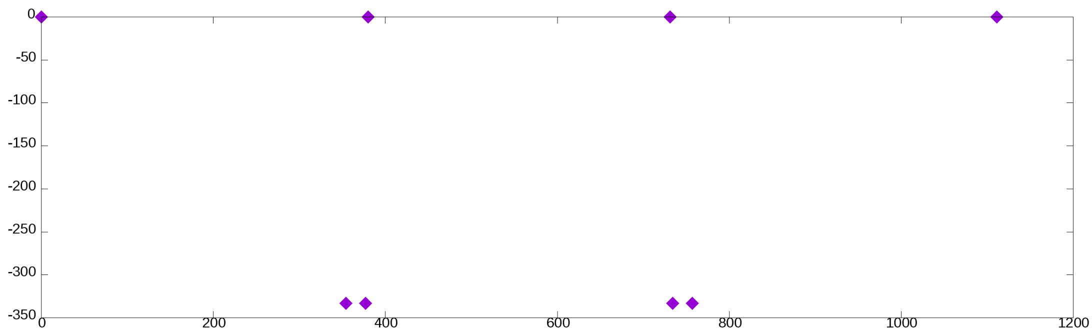

Рис. 2. *PIPS of cardinality 8 and diameter 1111*

Every cross set has characteristic 1; in [13], cross sets with only 2 points out of one of the lines that have two axes of symmetry are called *crabs*.

Definition 9. *A planar integral point set situated on two straight lines that are not parallel nor perpendicular, is called a scissors set.*

There is an important subclass of scissors sets.

Definition 10. *A scissors set with an axis of symmetry, which is the angle bisector for the straight lines, is called a pyramid set.*

Examples of pyramid sets are given in Section 4 on Fig. 25 and Fig. 26. We don't have any examples of pyramid sets with cardinality more than 9.

It still remains unknown whether a scissors set can have another axis of symmetry; on Fig. 4 the following scissors set with central symmetry is shown:

$$\sqrt{91}/1 * \{(0;0);(792;120);(-792;-120);\\ (\pm 256;0);(\pm 646;0);(\pm 693;0);(\pm 1152;0);(\pm 1242;0);(\pm 1682;0)\}$$

0.0.1 Other integral point sets

Definition 11. *A planar integral point set that is situated on a circle is called a circular point set.*

Circular sets are very important examples of integral point sets [20], [21], [22]; for example, the best known upper bound for $d(2,n)$ is attained on a circular set. Moreover, the first occurrence of integral point sets in modern literature (or at least the first one that the authors are aware of) is Anning's article from 1915 which focuses exactly on circular integral point sets [23]. For the most recent advances in circular IPS, we refer the reader to [24].

Definition 12. *A planar integral point set that is situated on the conjunction of a circle with its center, is called a centered-circular point set.*

Замечание 1. Every centered-circular point set has characteristic 1. Indeed, let $\mathcal{P} = \{M_0, M_1, M_2, M_3, ..., M_k\}$ be a centered-circular point set with M_0 being the center. By the Law of sines, in the triangle $M_1M_2M_3$

$$\sin\angle M_1M_2M_3 = \frac{|M_1M_3|}{2R}, \tag{6}$$

where $R = |M_0M_1|$ is the radius of the triangle's circumcircle. Then the area of the triangle $M_1M_2M_3$ is

$$S = \frac{1}{2}|M_2M_1|\cdot|M_2M_3|\cdot\sin\angle M_1M_2M_3 = \frac{1}{2}|M_2M_1|\cdot|M_2M_3|\cdot\frac{|M_1M_3|}{2|M_0M_1|} \in \mathbb{Q}. \tag{7}$$

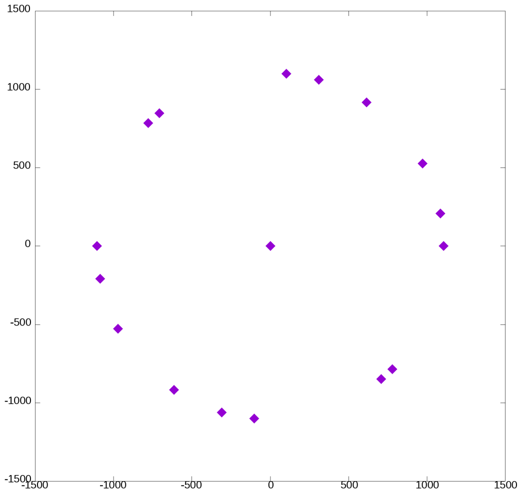

Рис. 3. *A centered-circular PIPS of cardinality 17 and diameter 2210*

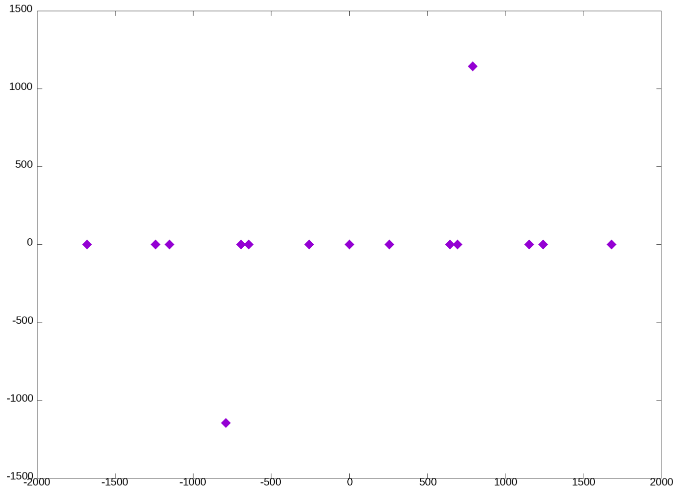

Рис. 4. *A scissors PIPS of cardinality 15 and diameter 3364*

On Fig. 3 the following centered-circular PIPS is shown:

$$\sqrt{1}/1105 * \{(0;0); (\pm1221025;0); (113953;1215696); (-113953;-1215696); (341887;1172184); (-341887;-1172184); (680225;1014000); (-680225;-1014000); (782977;-936936); (-782977;936936); (859775;-867000); (-859775;867000); (1073057;582624); (-1073057;-582624); (1198975;231000); (-1198975;-231000);\}$$

Most of the known planar integral point sets fall into these six classes; however, there are some sophisticated examples which do not [25], [26], [27]. A lower bound for the diameter of a PIPS with no collinear triples (so-called PIPS *in semi-general position*) is presented in [28]. It is important to note that the cardinality of an IPS in semi-general position is bounded not only by its diameter (i.e. the maximal distance between its points), but also by the minimal distance. Solymosi established such a bound [9]; in particular, he proved that if an IPS in semi-general position has distance 2, its cardinality cannot exceed 7. Recently, this result has been refined to 5 points [29].

3. BOUNDS BASED ON RAILS SETS

Let $i = \overline{1,k}$ denote the enumeration of all i from 1 to k. Theorem [8, Theorem 2.1] can be generalized as follows:

Теорема 2. *Let $\mathcal{P}$ be a planar integral point set consisting of $n-k$ points on the line l_1 and k points P_1, P_2, ..., P_k on a parallel line l_2 with distance r between l_1 and l_2. If there exist positive integers v, w_{ij}, with $f_{ij}^2 + v^2 = w_{ij}^2$ and $v < 2r$, where $i = \overline{1,k-1}$, $j = \overline{i+1,k}$, $|P_iP_j| = f_{ij}$, then*

$$d(m, n + k(m-2)) \leqslant \max(w_{1k}, \mathrm{diam}(\mathcal{P})) \quad (8)$$

Below we give the examples of planar integral point sets and the corresponding estimates of the function $d(m, n)$ for $n = 2m + k$, $3 \leqslant k \leqslant 36$.

Below f and w are those from Theorem 1.

- $\mathcal{P}_7 = \sqrt{315}/2 * \{(13, 1), (29, 1), (0, 0), (10, 0), (16, 0), (26, 0), (32, 0)\}$,
  $f = 8$, $v = 6$, $w = 10$, $\mathrm{diam}(\mathcal{P}_7) = 17$, which gives $d(m, 2m+3) \leqslant 17$.
- $\mathcal{P}_8 = \mathcal{P}_7 \cup \sqrt{315}/2 * \{(42, 0)\}$ (Figure 5),
  $f = 8$, $v = 6$, $w = 10$, $\mathrm{diam}(\mathcal{P}_8) = 21$, which gives $d(m, 2m+4) \leqslant 21$.
- $\mathcal{P}_9 = \sqrt{70}/1 * \{(\pm 44, 12), (-78, 0), (\pm 62, 0), (\pm 55, 0), (\pm 10, 0)\}$,
  $f = 88$, $v = 66$, $w = 110$, $\mathrm{diam}(\mathcal{P}_9) = 158$,
  which gives $d(m, 2m+5) \leqslant 158$.
- $\mathcal{P}_{10} = \mathcal{P}_9 \cup \sqrt{70}/1 * \{(78, 0)\}$ (Figure 6),
  $f = 88$, $v = 66$, $w = 110$, $\mathrm{diam}(\mathcal{P}_{10}) = 158$,
  which gives $d(m, 2m+6) \leqslant 158$.

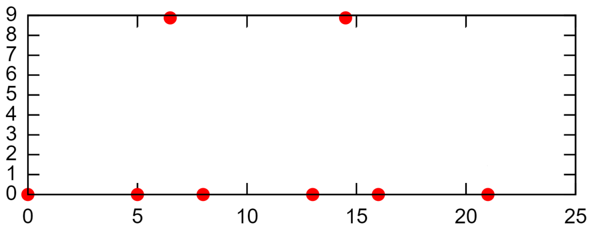

Рис. 5. *PIPS $\mathcal{P}_8$ of cardinality 8 and diameter 21*

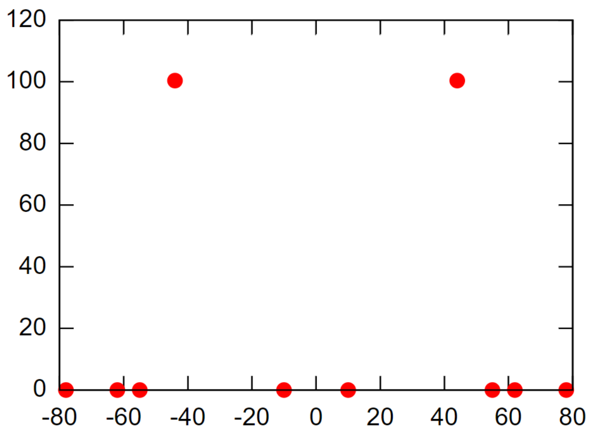

Рис. 6. *PIPS $\mathcal{P}_{10}$ of cardinality 10 and diameter 158*

- $\mathcal{P}_{11} = \sqrt{70}/2 * \{(\pm 99, 24), (\pm 207, 0), (\pm 145, 0), (\pm 63, 0), (\pm 25, 0), (-297, 0)\}$,
  $f = 99$, $v = 20$, $w = 101$, $\mathrm{diam}(\mathcal{P}_{11}) = 252$,
  which gives $d(m, 2m+7) \leqslant 252$.
- $\mathcal{P}_{12} = \mathcal{P}_{11} \cup \sqrt{70}/2 * \{(297, 0)\}$ (Figure 7),
  $f = 99$, $v = 20$, $w = 101$, $\mathrm{diam}(\mathcal{P}_{12}) = 297$,
  which gives $d(m, 2m+8) \leqslant 297$.
- $\mathcal{P}_{13} = \sqrt{19019}/2 * \{(\pm 200, 3), (\pm 873, 0), (\pm 615, 0), (\pm 377, 0), (\pm 215, 0), (\pm 23, 0),$ $(-1273, 0)\}$,
  $f = 200$, $v = 45$, $w = 205$, $\mathrm{diam}(\mathcal{P}_{13}) = 1073$,
  which gives $d(m, 2m+9) \leqslant 1073$.
- $\mathcal{P}_{14} = \mathcal{P}_{13} \cup \sqrt{19019}/2 * \{(1273, 0)\}$ (Figure 8),
  $f = 200$, $v = 45$, $w = 205$, $\mathrm{diam}(\mathcal{P}_{14}) = 1273$,
  which gives $d(m, 2m+10) \leqslant 1273$.

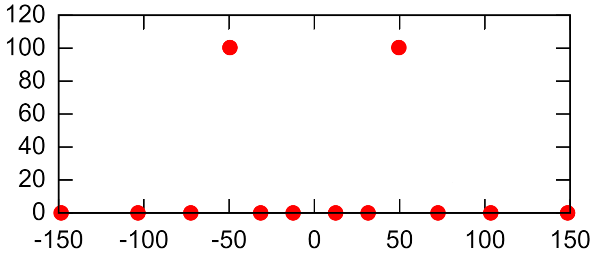

Рис. 7. *PIPS* $\mathcal{P}_{12}$ *of cardinality 12 and diameter 297*

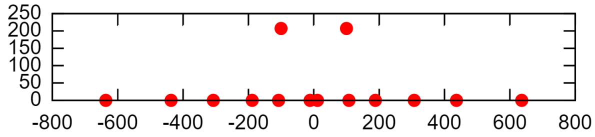

Рис. 8. *PIPS* $\mathcal{P}_{14}$ *of cardinality 14 and diameter 1273*

- $\mathcal{P}_{15} = \sqrt{385}/2 * \{(\pm 1105, 48), (\pm 1587, 0), (\pm 1269, 0), (\pm 763, 0), (\pm 623, 0), (\pm 529, 0), (\pm 339, 0), (-2189, 0)\}$,
  $f = 1105,\ v = 300,\ w = 1145,\ \operatorname{diam}(\mathcal{P}_{15}) = 1888,$
  which gives $d(m, 2m + 11) \leqslant 1888.$
- $\mathcal{P}_{16} = \mathcal{P}_{15} \cup \sqrt{385}/2 * \{(2189, 0)\}$ (Figure 9),
  $f = 1105,\ v = 300,\ w = 1145,\ \operatorname{diam}(\mathcal{P}_{16}) = 2189,$
  which gives $d(m, 2m + 12) \leqslant 2189.$

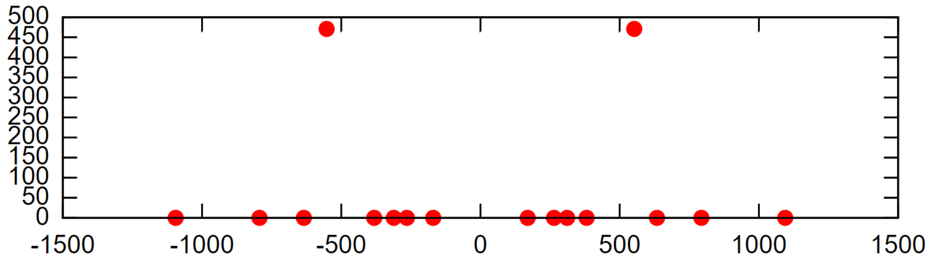

Рис. 9. *PIPS* $\mathcal{P}_{16}$ *of cardinality 16 and diameter 2189*

- $\mathcal{P}_{17} = \sqrt{154}/1 * \{(\pm 874, 60), (\pm 1376, 0), (\pm 1036, 0), (\pm 899, 0), (\pm 849, 0), (\pm 613, 0), (\pm 576, 0), (\pm 100, 0), (-1848, 0)\}$,
  $f = 1748,\ v = 336,\ w = 1780,\ \operatorname{diam}(\mathcal{P}_{17}) = 3224,$
  which gives $d(m, 2m + 13) \leqslant 3224.$
- $\mathcal{P}_{18} = \mathcal{P}_{17} \cup \sqrt{154}/1 * \{(1848, 0)\}$ (Figure 10),
  $f = 1748,\ v = 336,\ w = 1780,\ \operatorname{diam}(\mathcal{P}_{18}) = 3696,$
  which gives $d(m, 2m + 14) \leqslant 3696.$

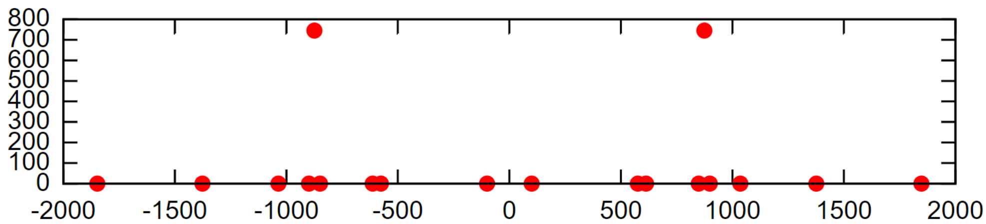

Рис. 10. *PIPS* $\mathcal{P}_{18}$ *of cardinality 18 and diameter 3696*

- $\mathcal{P}_{19} = \mathcal{P}_{18} \cup \sqrt{154}/1 * \{(-3293, 0)\}$,
  $f = 1748,\ v = 336,\ w = 1780,\ \operatorname{diam}(\mathcal{P}_{19}) = 5141,$
  which gives $d(m, 2m + 15) \leqslant 5141.$

- $\mathcal{P}_{20} = \mathcal{P}_{19} \cup \sqrt{154}/1 * \{(3293, 0)\}$ (Figure 11),
  $f = 1748,\ v = 336,\ w = 1780,\ \mathrm{diam}(\mathcal{P}_{20}) = 6586$,
  which gives $d(m, 2m + 16) \leqslant 6586$.

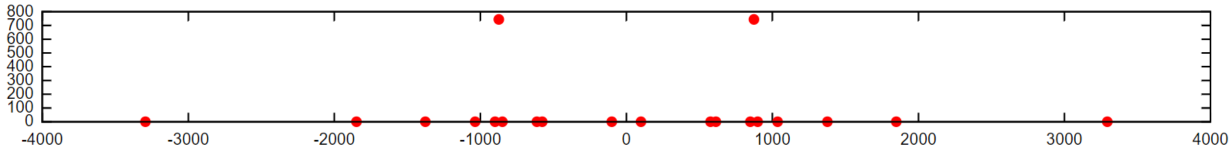

Рис. 11. *PIPS $\mathcal{P}_{20}$ of cardinality 20 and diameter 6586*

- $\mathcal{P}_{21} = \sqrt{154}/1 * \{(\pm 2622, 180), (\pm 5544, 0), (\pm 4128, 0), (\pm 3108, 0), (\pm 2697, 0),$
  $(\pm 2547, 0), (\pm 1839, 0), (\pm 1728, 0), (\pm 904, 0), (\pm 300, 0), (-6148, 0)\}$,
  $f = 5244,\ v = 1008,\ w = 5340,\ \mathrm{diam}(\mathcal{P}_{21}) = 11692$,
  which gives $d(m, 2m + 17) \leqslant 11692$.
- $\mathcal{P}_{22} = \mathcal{P}_{21} \cup \sqrt{154}/1 * \{(6148, 0)\}$,
  $f = 5244,\ v = 1008,\ w = 5340,\ \mathrm{diam}(\mathcal{P}_{22}) = 12296$,
  which gives $d(m, 2m + 18) \leqslant 12296$.
- $\mathcal{P}_{23} = \mathcal{P}_{22} \cup \sqrt{154}/1 * \{(-9879, 0)\}$,
  $f = 5244,\ v = 1008,\ w = 5340,\ \mathrm{diam}(\mathcal{P}_{23}) = 16027$,
  which gives $d(m, 2m + 19) \leqslant 16027$.
- $\mathcal{P}_{24} = \mathcal{P}_{23} \cup \sqrt{154}/1 * \{(9879, 0)\}$ (Figure 12),
  $f = 5244,\ v = 1008,\ w = 5340,\ \mathrm{diam}(\mathcal{P}_{24}) = 19758$,
  which gives $d(m, 2m + 20) \leqslant 19758$.
- $\mathcal{P}_{25} = \sqrt{154}/1 * \{(\pm 5244, 360), (\pm 12296, 0), (\pm 11088, 0), (\pm 8579, 0), (\pm 8256, 0),$
  $(\pm 6216, 0), (\pm 5394, 0), (\pm 5094, 0), (\pm 3678, 0), (\pm 3456, 0), (\pm 1808, 0), (\pm 600, 0),$
  $(-19758, 0)\}$,
  $f = 10488,\ v = 1015,\ w = 10537,\ \mathrm{diam}(\mathcal{P}_{25}) = 32054$,
  which gives $d(m, 2m + 21) \leqslant 32054$.

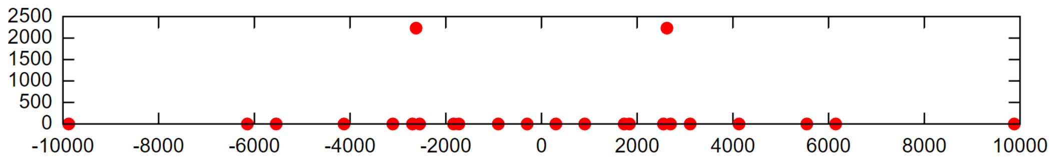

Рис. 12. *PIPS $\mathcal{P}_{24}$ of cardinality 24 and diameter 19758*

- $\mathcal{P}_{26} = \mathcal{P}_{25} \cup \sqrt{154}/1 * \{(19758, 0)\}$ (Figure 13),
  $f = 10488,\ v = 1015,\ w = 10537,\ \mathrm{diam}(\mathcal{P}_{26}) = 39516$,
  which gives $d(m, 2m + 22) \leqslant 39516$.
- $\mathcal{P}_{27} = \sqrt{154}/1 * \{(\pm 36708, 2520), (\pm 116058, 0), (\pm 86072, 0), (\pm 77616, 0), (\pm 60053, 0),$
  $(\pm 57792, 0), (\pm 43512, 0), (\pm 37758, 0), (\pm 35658, 0), (\pm 25746, 0), (\pm 24192, 0),$
  $(\pm 12656, 0), (\pm 4200, 0), (-138306, 0)\}$,
  $f = 73416,\ v = 2710,\ w = 73466,\ \mathrm{diam}(\mathcal{P}_{27}) = 254364$,
  which gives $d(m, 2m + 23) \leqslant 254364$.

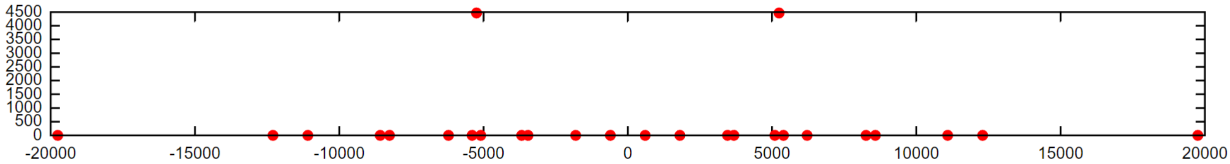

Рис. 13. *PIPS $\mathcal{P}_{26}$ of cardinality 26 and diameter 39516*

- $\mathcal{P}_{28} = \mathcal{P}_{27} \cup \sqrt{154}/1 * \{(138306, 0)\}$,
  $f = 73416,\ v = 2710,\ w = 73466,\ \mathrm{diam}(\mathcal{P}_{28}) = 276612$,
  which gives $d(m, 2m + 24) \leqslant 276612$.
- $\mathcal{P}_{29} = \mathcal{P}_{28} \cup \sqrt{154}/1 * \{(-199863, 0)\}$,
  $f = 73416,\ v = 2710,\ w = 73466,\ \mathrm{diam}(\mathcal{P}_{29}) = 338169$,
  which gives $d(m, 2m + 25) \leqslant 338169$.
- $\mathcal{P}_{30} = \mathcal{P}_{29} \cup \sqrt{154}/1 * \{(199863, 0)\}$ (Figure 14),
  $f = 73416,\ v = 2710,\ w = 73466,\ \mathrm{diam}(\mathcal{P}_{30}) = 399726$,
  which gives $d(m, 2m + 26) \leqslant 399726$.

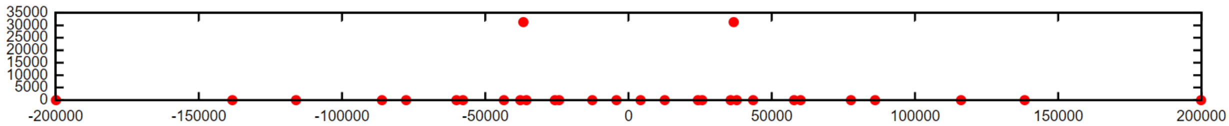

Рис. 14. *PIPS $\mathcal{P}_{30}$ of cardinality 30 and diameter 399726*

- $\mathcal{P}_{31} = \sqrt{154}/1 * \{(\pm 580290, 0), (\pm 430360, 0), (\pm 388080, 0), (\pm 300265, 0),$
  $(\pm 288960, 0), (\pm 217560, 0), (\pm 188790, 0), (\pm 183540, 12600), (\pm 178290, 0),$
  $(\pm 128730, 0), (\pm 120960, 0), (\pm 63280,), (\pm 21000, 0), (\pm 11224, 0),$
  $(-1033912, 0), (-999315, 0), (-691530, 0)\}$,
  $f = 367080,\ v = 3206,\ w = 367094,\ \mathrm{diam}(\mathcal{P}_{31}) = 1614202$,
  which gives $d(m, 2m + 27) \leqslant 1614202$.
- $\mathcal{P}_{32} = \mathcal{P}_{31} \cup \sqrt{154}/1 * \{(691530, 0)\}$ (Figure 15),
  $f = 367080,\ v = 3206,\ w = 367094,\ \mathrm{diam}(\mathcal{P}_{32}) = 1725442$,
  which gives $d(m, 2m + 28) \leqslant 1725442$.

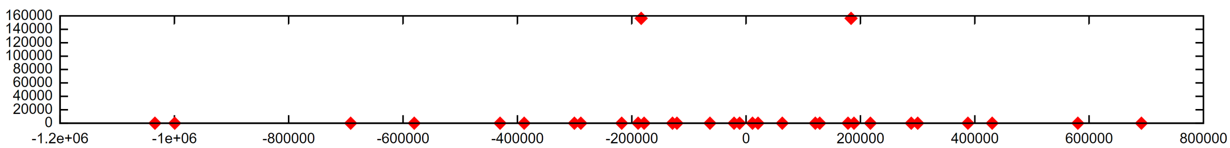

Рис. 15. *PIPS $\mathcal{P}_{32}$ of cardinality 32 and diameter 1725442*

- $\mathcal{P}_{33} = \mathcal{P}_{32} \cup \sqrt{154}/1 * \{(999315, 0)\}$,
  $f = 367080,\ v = 3206,\ w = 367094,\ \mathrm{diam}(\mathcal{P}_{33}) = 2033227$,
  which gives $d(m, 2m + 29) \leqslant 2033227$.
- $\mathcal{P}_{34} = \mathcal{P}_{33} \cup \sqrt{154}/1 * \{(1033912, 0)\}$ (Figure 16),
  $f = 367080,\ v = 3206,\ w = 367094,\ \mathrm{diam}(\mathcal{P}_{34}) = 2067824$,
  which gives $d(m, 2m + 30) \leqslant 2067824$.

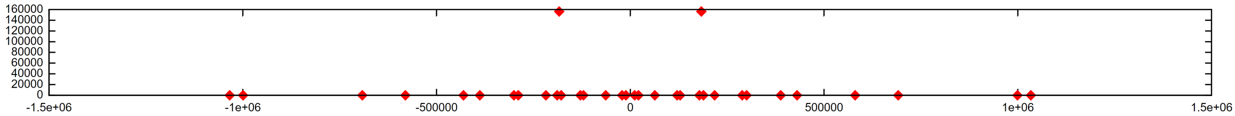

Рис. 16. *PIPS $\mathcal{P}_{34}$ of cardinality 34 and diameter 2067824*

- $\mathcal{P}_{35} = \sqrt{154}/1 * \{(\pm 3997260, 0), (\pm 2766120, 0), (\pm 2321160, 0), (\pm 1721440, 0), (\pm 1552320, 0), (\pm 1201060, 0), (\pm 1155840, 0), (\pm 870240, 0), (\pm 755160, 0), (\pm 734160, 50400), (\pm 713160, 0), (\pm 514920, 0), (\pm 483840, 0), (\pm 308175, 0), (\pm 253120, 0), (\pm 84000, 0), (\pm 44896, 0), (-4135648, 0)\}$,
  $f = 1468320,\ v = 12824,\ w = 1468376,\ \mathrm{diam}(\mathcal{P}_{35}) = 8132908,$
  which gives $d(m, 2m+31) \leqslant 8132908$.
- $\mathcal{P}_{36} = \mathcal{P}_{35} \cup \sqrt{154}/1 * \{(4135648, 0)\}$ (Figure 17),
  $f = 1468320,\ v = 12824,\ w = 1468376,\ \mathrm{diam}(\mathcal{P}_{36}) = 8271296,$
  which gives $d(m, 2m+32) \leqslant 8271296$.

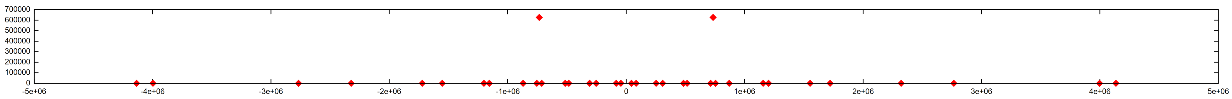

Рис. 17. *PIPS $\mathcal{P}_{36}$ of cardinality 36 and diameter 8271296*

- $\mathcal{P}_{37} = \sqrt{154}/1 * \{(\pm 12991095, 0), (\pm 8989890, 0), (\pm 7543770, 0), (\pm 5594680, 0), (\pm 5045040, 0), (\pm 3903445, 0), (\pm 3756480, 0), (\pm 3694845, 0), (\pm 2828280, 0), (\pm 2454270, 0), (\pm 2386020, 163800), (\pm 2317770, 0), (\pm 1673490, 0), (\pm 1572480, 0), (\pm 822640, 0), (\pm 484680, 0), (\pm 273000, 0), (\pm 145912, 0), (-13440856, 0)\}$,
  $f = 4772040,\ v = 41678,\ w = 4772222,\ \mathrm{diam}(\mathcal{P}_{37}) = 26431951,$
  which gives $d(m, 2m+33) \leqslant 26431951$.
- $\mathcal{P}_{38} = \mathcal{P}_{37} \cup \sqrt{154}/1 * \{(13440856, 0)\}$ (Figure 18),
  $f = 4772040,\ v = 41678,\ w = 4772222,\ \mathrm{diam}(\mathcal{P}_{38}) = 26881712,$
  which gives $d(m, 2m+34) \leqslant 26881712$.

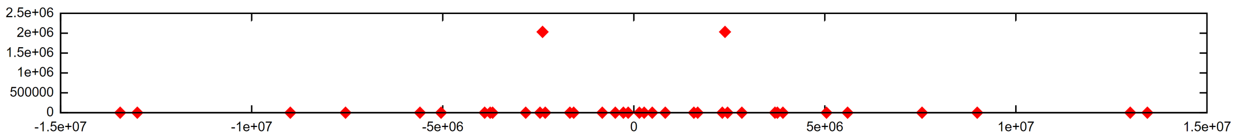

Рис. 18. *PIPS $\mathcal{P}_{38}$ of cardinality 38 and diameter 26881712*

- $\mathcal{P}_{39} = \sqrt{154}/1 * \{(\pm 51964380, 0), (\pm 35959560, 0), (\pm 30175080, 0), (\pm 22378720, 0), (\pm 20180160, 0), (\pm 15613780, 0), (\pm 15025920, 0), (\pm 14779380, 0), (\pm 11313120, 0), (\pm 9817080, 0), (\pm 9544080, 655200), (\pm 9271080, 0), (\pm 6693960, 0), (\pm 6289920, 0), (\pm 4006275, 0), (\pm 3290560, 0), (\pm 1938720, 0), (\pm 1092000, 0), (\pm 583648, 0), (-53763424, 0)\}$,
  $f = 19088160,\ v = 8738,\ w = 19088162,\ \mathrm{diam}(\mathcal{P}_{39}) = 105727804,$
  which gives $d(m, 2m+35) \leqslant 105727804$.

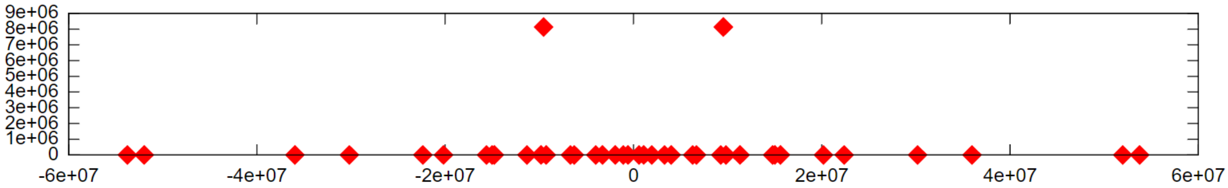

Рис. 19. *PIPS of cardinality 40 and diameter 107526848*

- $\mathcal{P}_{40} = \mathcal{P}_{39} \cup \sqrt{154}/1 * \{(53763424, 0)\}$ (Figure 19),

  $f = 19088160$, $v = 8738$, $w = 19088162$, $\operatorname{diam}(\mathcal{P}_{40}) = 107526848$,

  which gives $d(m, 2m + 36) \leqslant 107526848$.

Замечание 2. In all the examples above, the maximum in the right-hand side of (5) is attained on the diameter of the system. The following example shows that the maximum can be attained on w:

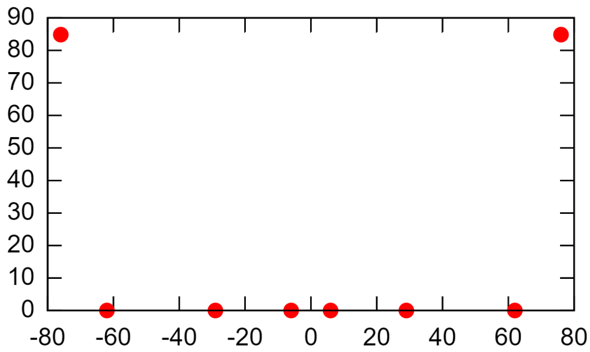

Рис. 20. *PIPS of cardinality 8 and diameter 162*

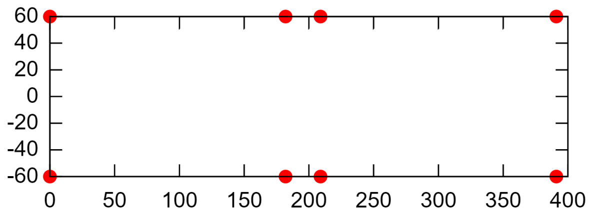

Рис. 21. *PIPS of cardinality 8 and diameter 409*

- $\mathcal{P} = \sqrt{2}/1 * \{(\pm 76, 60), (\pm 62, 0), (\pm 29, 0), (\pm 6, 0)\}$ (Figure 20),

  $f = 152$, $v = 114$, $w = 190$, $\operatorname{diam}(\mathcal{P}) = 162$.

  We can obtain the estimate $d(m, 2m + 2) \leqslant 190$ (which is weaker than the one given in (3)).

Guy gives [30, D 20] a PIPS of 8 points located on two parallel lines. The points of the set form a rectangle (Figure 22):

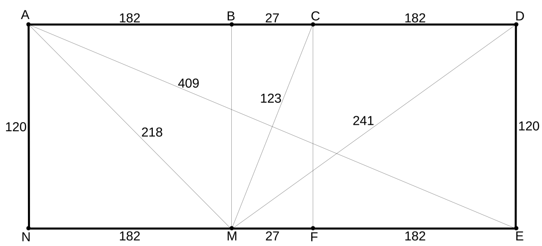

Рис. 22. *Distances*

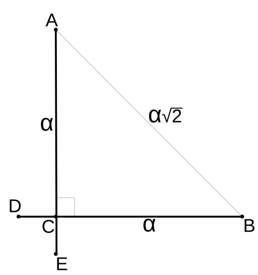

Рис. 23. *Contradiction*

- $\mathcal{P} = \sqrt{1}/1 * \{(0, \pm 60), (182, \pm 60), (209, \pm 60), (391, \pm 60)\}$ (Figure 21), where $n = 8$, $\mathrm{diam}(\mathcal{P}) = 409$. Using the "blowing up" construction, we obtain

$$d(m, 4m) \leqslant 409 \tag{9}$$

Figure 24 shows an example for $m = 3$.

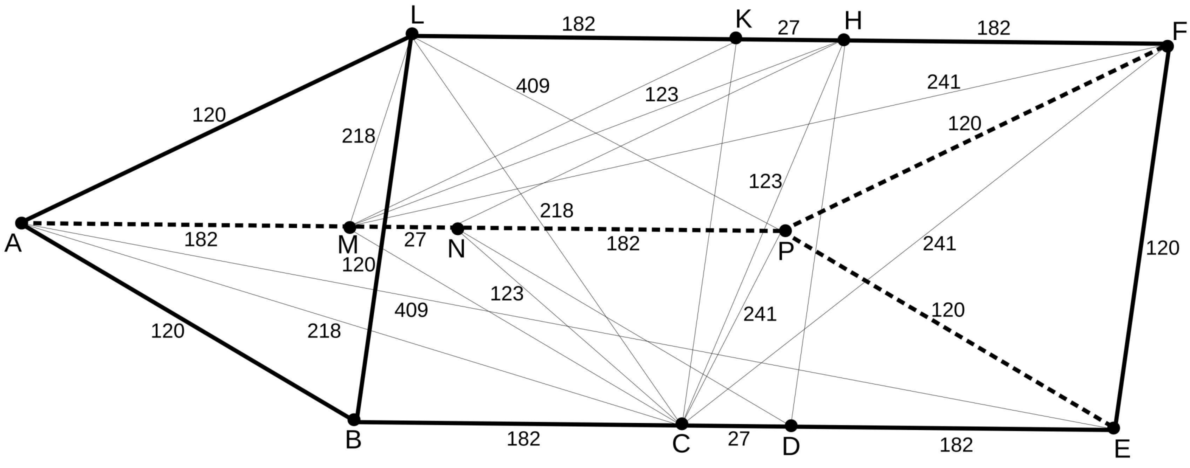

Рис. 24. *IPS of cardinality 12 and diameter 409*

4. BOUNDS BASED ON PYRAMID SETS

Теорема 3. *Let $\mathcal{P}$ be a planar integral point set consisting of k points on the line l_1 and k points on the line l_2. Besides, these points are symmetric with respect to one of the bisectors of the angles formed by the intersection of lines l_1 and l_2, then*

$$d(m, (k - \alpha)m + \alpha) \leqslant \mathrm{diam}(\mathcal{P}), \tag{10}$$

where

$$\alpha = \begin{cases} 1, \textit{when the intersection point} \in \mathcal{P}, \\ 0, \textit{when the intersection point} \notin \mathcal{P}. \end{cases}$$

Замечание 3. The angles can be acute or obtuse. Figure 23 shows that the intersection angle of the lines l_1 and l_2 cannot be equal to $\pi/2$.

Below we give the examples of planar integral point sets and the corresponding estimates of the function $d(m, n)$ for $n = 3m + 1$ and $n = 4m + 1$.

- $\mathcal{P} = \sqrt{3}/2 * \{(\pm 56, 0), (14, 0), (-34, 0), (-10, 24), (-21, 35), (35, -21)\}$ (Figure 25), where $n = 7$, $\mathrm{diam}(\mathcal{P}) = 56$. Using the "blowing up" construction, we obtain

$$d(m, 3m + 1) \leqslant 56 \tag{11}$$

  The resulting estimate improves the estimate for $d(m, 3m)$, which is presented in [6]. Figure 27 shows an example for $m = 3$.
- $\mathcal{P} = \sqrt{39}/8 * \{(\pm 5040, 0), (1911, 315), (2352, 0), (944, 0), (336, 0), (2940, -420), (2044, 220), (735, 1155)\}$ (Figure 26),

  where $n = 9$, $\mathrm{diam}(\mathcal{P}) = 1260$. Using the "blowing up" construction, we obtain

$$d(m, 4m + 1) \leqslant 1260 \tag{12}$$

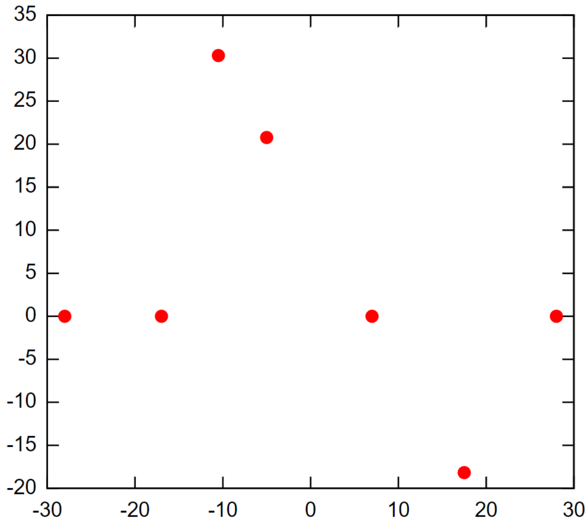

Рис. 25. *PIPS of cardinality 7 and diameter 56*

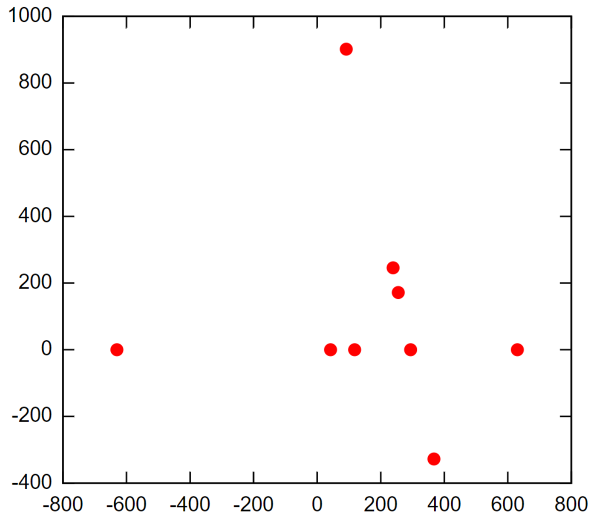

Рис. 26. *PIPS of cardinality 9 and diameter 1260*

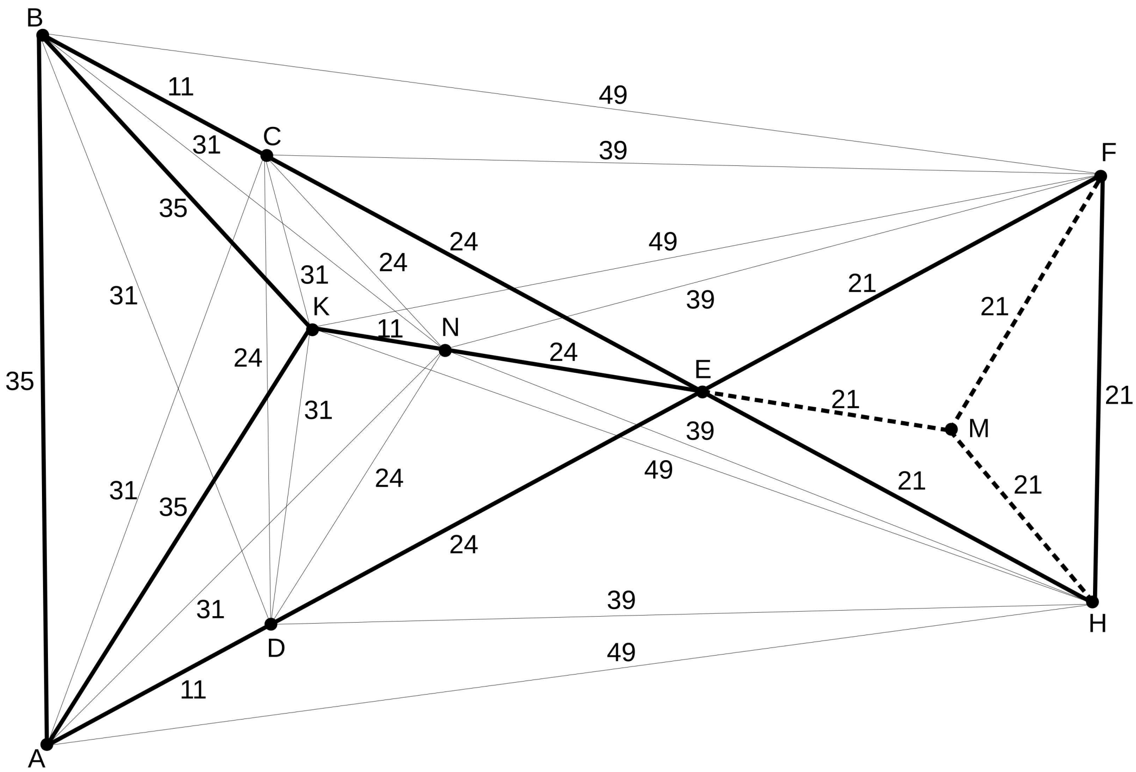

Рис. 27. *IPS of cardinality 10 and diameter 56*

Figure 28 shows an example for $m = 3$.

However, we have to admit that both bounds which employ pyramid sets are indeed weaker than the one

$$d(m, m^2 + m) \leqslant 17 \tag{13}$$

proved in [8], so the constructions of bounds based on pyramid sets tend to be only for the conceptual purpose.

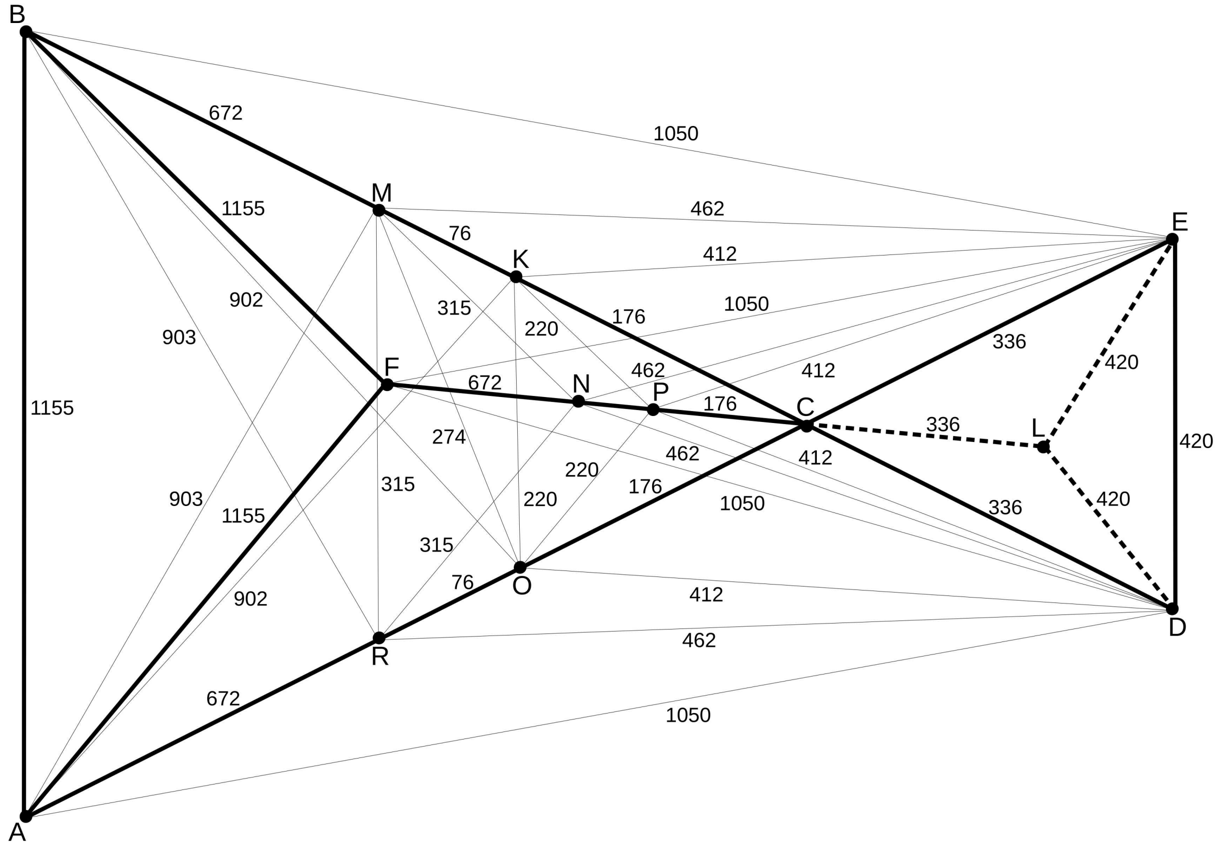

Рис. 28. *IPS of cardinality 13 and diameter 1260*

5. FINAL REMARKS

All the discussed PIPSs were obtained through a combination of computer search and intuition of the authors; so, the further search may lead to better bounds employing the same constructions.

There is still no general construction for a rails or scissors PIPS of arbitrary cardinality. For rails PIPSs, we can conjecture that there exists a set of arbitrary cardinality, with 2 points on one line and all the rest on the other; for an example of cardinality 106, we refer the reader to [31]. On the other hand, it is still unknown whether there are any rails PIPSs with 4 and 5 points on the lines.

As for today, we have found pyramid PIPS of cardinality at most 9.

The source code can be obtained at https://gitlab.com/Nickkolok/ips-algo . The preprint of this article is arXiv:1909.10386 [math.CO]

The authors thank A. S. Usachev for his help in polishing the text of the paper.

REFERENCES

1. Ascher, K. The Erdős–Ulam problem, Lang's conjecture and uniformity / K. Ascher, L. Braune, A. Turchet // Bulletin of the London Mathematical Society. — 2020. — V. 52, № 6. — P. 1053–1063.

2. Solymosi, J. On a question of Erdős and Ulam / J. Solymosi, F. De Zeeuw // Discrete & Computational Geometry. — 2010. — V. 43, № 2. — P. 393–401.

3. Kurz, S. Enumeration of Integral Tetrahedra / S. Kurz // Journal of Integer Sequences. — 2007. — V. 10, № 2. — P. 3.

4. Anning, N. H. Integral distances / N. H. Anning, P. Erdős // Bulletin of the American Mathematical Society. — 1945. — V. 51, № 8. — P. 598–600.

5. Erdős, P. Integral distances / P. Erdős // Bulletin of the American Mathematical Society. — 1945. — V. 51, № 12. — P. 996.

6. Kemnitz, A. Punktmengen mit ganzzahligen Abständen / A. Kemnitz. — 1988.

7. Kurz, S. On the characteristic of integral point sets in $\mathbb{E}^m$ / S. Kurz // Australasian Journal of Combinatorics. — 2006. — V. 36. — P. 241–248.

8. Kurz, S. Bounds for the minimum diameter of integral point sets / S. Kurz, R. Laue // Australasian Journal of Combinatorics. — 2007. — V. 39. — P. 233–240.

9. Solymosi, J. Note on integral distances / J. Solymosi // Discrete & Computational Geometry. — 2003. — V. 30, № 2. — P. 337–342.

10. Avdeev, N. On existence of integral point sets and their diameter bounds / N. Avdeev // Australasian Journal of Combinatorics. — 2020. — V. 77, no. 1. — P. 100–116.

11. Nozaki, H. Lower bounds for the minimum diameter of integral point sets / H. Nozaki // Australasian Journal of Combinatorics. — 2013. — V. 56. — P. 139–143.

12. Авдеев, Н. Н. Множества точек с целочисленными расстояниями на плоскости и в евклидовом пространстве / Н. Н. Авдеев, Е. М. Семёнов // Математический форум (Итоги науки. Юг России). — 2018. — С. 217–236.

13. Antonov, A. R. Maximal integral point sets over $\mathbb{Z}^2$ / A. R. Antonov, S. Kurz // International Journal of Computer Mathematics. — 2008. — V. 87, № 12. — P. 2653–2676.

14. Kurz, S. On the minimum diameter of plane integral point sets / S. Kurz, A. Wassermann // Ars Combinatoria. — 2011. — V. 101. — P. 265–287.

15. Zvolinsky, R. E. Facher integral point sets with particular distances of arbitrary cardinality / R. E. Zvolinsky // Актуальные проблемы прикладной математики, информатики и механики. — 2021. — P. 668–674.

16. Лушина, Е. А. О характеристике плоских множеств с целочисленными расстояниями с рёбрами 1 и 2 / Е. А. Лушина, Н. Н. Авдеев // Вестник факультета прикладной математики, информатики и механики. — 2021. — V. 15. — P. 117–123.

17. Авдеев, Н. Н. On integral point sets in special position / Н. Н. Авдеев // Некоторые вопросы анализа, алгебры, геометрии и математического образования: материалы международной молодежной научной школы «Актуальные направления математического анализа и смежные вопросы». — 2018. — Т. 8. — С. 5–6.

18. Авдеев, Н. Н. Об отыскании целоудалённых множеств специального вида / Н. Н. Авдеев // Актуальные проблемы прикладной математики, информатики и механики — сборник трудов Международной научной конференции. — Научно-исследовательские публикации, 2018. — С. 492–498.

19. Лушина, Е. А. О длине основания рельсового целоудалённого множества / Е. А. Лушина // Современные проблемы математики и её приложений. — Институт математики и механики УрО РАН им. Н.Н. Красовского, 2025.

20. Harborth, H. An upper bound for the minimum diameter of integral point sets / H. Harborth, A. Kemnitz, M. Möller // Discrete & Computational Geometry. — 1993. — V. 9, № 4. — P. 427–432.

21. Piepmeyer, L. The maximum number of odd integral distances between points in the plane / L. Piepmeyer // Discrete & Computational Geometry. — 1996. — V. 16, № 1. — P. 113–115.

22. Bat-Ochir, G. On the number of points with pairwise integral distances on a circle / G. Bat-Ochir // Discrete Applied Mathematics. — 2018. — V. 254. — P. 17–32.

23. Anning, N. Discussions: Relating to a Geometric Representation of Integral Solutions of Certain Quadratic Equations / N. Anning // The American Mathematical Monthly. — 1915. — V. 22, № 9. — P. 321–321.

24. Undrakh, B. Refinement of the upper bound of the radius of a circular integral points set / B. Undrakh, G. Bat-Ochir // Gulf Journal of Mathematics. — 2022. — V. 12, № 1. — P. 28–40.

25. Kreisel, T. There are integral heptagons, no three points on a line, no four on a circle / T. Kreisel, S. Kurz // Discrete & Computational Geometry. — 2008. — V. 39, № 4. — P. 786–790.

26. Constructing 7-clusters / S. Kurz [et al.] // Serdica Journal of Computing. — 2014. — V. 8, № 1. — P. 47–70.

27. Avdeev, N. On particular examples of planar integral point sets and their classification / N. Avdeev, E. Momot, A. Zvolinsky. — arXiv: http://arxiv.org/abs/2102.12462 [math.CO]

28. Avdeev, N. On diameter bounds for planar integral point sets in semi-general position / N. Avdeev // Chebyshevskii Sbornik. — 2019. — V. 22, № 4. — P. 343–350.

29. Лушина, Е. А. О мощности плоского целоудаленного множества с ребром 2 / Е. А. Лушина, В. С. Бриндюк // 74-я Международная студенческая научно-техническая конференция. Материалы. Астрахань. — 2024. — С. 741–743.

30. Guy, R. Unsolved problems in number theory. Vol. 1 / R. Guy. — Springer Science & Business Media, 2013.

31. Momot, E. A. An example of rails integral point set with cardinality 106 / E. A. Momot, R. E. Zvolinskiy, A. E. Zvolinskiy // Прикладная математика и фундаментальная информатика. — 2022. — С. 114–115.

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