UDK 517.9

CAUCHY CONDITION FOR BARENBLATT-ZHELTOV-KOCHINA EQUATIONS IN QUASI-BANACH SPACES

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Поступила в редакцию 31.10.2020 г.

Abstract: In this article, a Barenblatt-Zheltov-Kochina model as an example on linear Sobolev type equations , is introduced with a Cauchy condition in quasi-Banach spaces. In this model, we are used quasi-Laplace operators which are defined on quasi-Sobolev spaces.

Key words and phrases: Quasi-Banach spaces, Quasi-Sobolev spaces, Laplace' Quasi-operator, Cauchy condition.

Условие Коши для Уравнений Баренблатта-Желтова-Кочиной в Квазибанаховых Пространствах

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Аннотация: В этой статье, модель Баренблатта-Желтова-Кочиной в качестве примера линейных уравнений соболевского типа вводится с условием Коши в квазибанаховых пространствах. В этой модели используются квазилапласовы операторы, определенные на квазисоболевских пространствах.

Ключевые слова: квазибанаховы пространства, квазисоболевы пространства, квазиоператор Лапласа, условием Коши.

INTRODUCTION

Quasi-normed space $(\mathfrak{U}, {}_{\mathfrak{U}}\|\cdot\|)$ is a vector space $\mathfrak{U}$ over a field $\mathbb{F}$ (a set of real or complex numbers)with *quasi-norm* ${}_{\mathfrak{U}}\|\cdot\|$, which differs from the norm only by « inequality » : $\forall u,v \in \mathfrak{U}$ ${}_{\mathfrak{U}}\|u+v\| \leqslant C({}_{\mathfrak{U}}\|u\| + {}_{\mathfrak{U}}\|v\|)$, where a constant $C \geqslant 1$. If $C = 1$, then the quasi-norm becomes a norm, and the quasi-normed space turns into a normed space. Generally, a quasi-normed space is not normed space but metrizable [1], Lemma 3.10.1, then it is topological linear spaces and the concepts of fundamental sequence and completeness are correct. A complete quasi-normed space $\mathfrak{U}$ is called quasi-Banach space[1,2] .

A monotonically increasing sequence $\{\lambda_k\} \subset \mathbf{R}_+$, $k \in \mathbf{N}$ such that $\lim\limits_{k\to\infty} \lambda_k = +\infty$ was used to construct quasi-Sobolev spaces l_q^m:
$l_q^m = \left\{ u = \{u_k\} : \sum\limits_{k=1}^{\infty} \left(\lambda_k^{\frac{m}{2}} |u_k|\right)^q < +\infty \right\}$. $q \in (0, +\infty), m \in \mathbb{R}$, and then to define quasi- Laplace operator $\Lambda u = \lambda_k u_k$ on these spaces with its inverse $\Lambda^{-1} u = \{\lambda_k^{-1} u_k\}$-*quasi Green's operator.* We were proved l_q^m as a quasi-Banach space, and Λ -toplinear isomorphism operator [3,4,5].

A Barenblatt-Zheltov-Kochina equation is the most famous non-classical equation simulating the processes of filtration, thermal conductivity and moisture, which was studied in a concept of Banach space and has received much attention in a lot of the references [6,7,8].

In this article, we introduce non homogeneous linear Sobolev type equation with Cauchy condition in quasi-Banach spaces, and take model Barenblatt-Zheltov-Kochina as an example on it.

1.Relatively $p-$Bounded Operators

Let $\mathfrak{U}$ and $\mathfrak{F}$ be quasi-Banach spaces. A linear operator $L : \mathfrak{U} \to \mathfrak{F}$ is called continuous if $\lim\limits_{k\to\infty} Lu_k = L\left(\lim\limits_{k\to\infty} u_k\right)$ for any sequence $\{u_k\} \subset \mathfrak{U}$, converging in $\mathfrak{U}$; and is called bounded if it maps bounded sets to bounded sets.

A continuous linear operator L is called toplinear isomorphism if there exists an inverse operator $L^{-1} : \mathfrak{F} \to \mathfrak{U}$, which is also continuous. The space of linear bounded operators $\mathcal{L}(\mathfrak{U};\mathfrak{F})$ is quasi-Banach with the quasi-norm ${}_{\mathcal{L}(\mathfrak{U};\mathfrak{F})}\|L\| = \sup\limits_{\mathfrak{U}\|u\|=1} {}_{\mathfrak{F}}\|Lu\|$.

Theorem 1. For every $q \in (0, +\infty)$, $m \in \mathbb{R}$, l_q^m be a quasi-Banach space with ${}_q\|u\|_m = \left(\sum\limits_{k=1}^{\infty} \lambda_k^{mq/2}|u_k|^q\right)^{1/q}$.

Remark. We observe that the spaces ℓ_q^m do not depend on choice of a sequence $\{\lambda_k\}$, and there are dense and continuous embedding $\ell_q^n \hookrightarrow \ell_q^m$ for $n \geqslant m$ and $q \in \mathbb{R}_+$. Also, we note that a constant $\mathrm{C} = 2^{\frac{1}{q}-1}$ when $q \in (0,1)$, while $\mathrm{C} = 1$ when $q \in [1,\infty)$.

Theorem 2. For every $q \in (0, +\infty)$, $m \in \mathbb{R}$, a quasi-Laplace operator $\Lambda : l_q^{m+2} \to l_q^m$ is a toplinear isomorphism operator.

Let operators $L,M \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, then a L-resolvent set:

$\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})\}$ and a L-spectrum $\sigma^L(M) = \mathbb{C}\backslash\rho^L(M)$ of an operator M. Suppose $\rho^L(M) \neq \varphi$, then operator-functions: $(\mu L - M)^{-1}$,$R_\mu^L(M) = (\mu L - M)^{-1}L$ and $L_\mu^L(M) = L(\mu L - M)^{-1}$ are called L resolvent, right and left L resolvent of an operator M respectively. We observe that:$\forall,\lambda,\mu \in \mathbb{C}$,

$$(\mu L - M)^{-1}(\lambda L - M)^{-1} = \mathbb{I} + (\lambda - \mu)R_\lambda^L(M) \tag{1}$$

$$R_\lambda^L(M) - R_\mu^L(M) = (\mu - \lambda)R_\mu^L(M)R_\lambda^L(M), \tag{2}$$

An operator M is said to be *spectrally bounded* with respect to an operator L (shortly, $M(L,\sigma)$-*bounded*) if

$$\exists a \in \mathbb{R}_+ \ \forall \mu \in \mathbb{C}\ (|\mu| > a) \ \Rightarrow \ \left(\mu \in \rho^L(M)\right).$$

Let an operator $M(L,\sigma)$-bounded, and a contour:

$\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$. Consider integrals of the type F. Rissa

$$P = \frac{1}{2\pi i}\int\limits_\gamma R_\mu^L(M)d\mu, \qquad Q = \frac{1}{2\pi i}\int\limits_\gamma L_\mu^L(M)d\mu.$$

Lemma 1 . Let an operator $M(L,\sigma)$-bounded, then operators $P \in \mathcal{L}(\mathfrak{U})$ $(\equiv \mathcal{L}(\mathfrak{U};\mathfrak{U}))$ and $Q \in \mathcal{L}(\mathfrak{F})(\equiv \mathcal{L}(\mathfrak{F};\mathfrak{F}))$ are projectors.

Proof. Take a contour $\acute{\gamma} = \{\lambda \in \mathbb{C} : |\lambda| = \acute{r} > r\}$. According to the analyticity of integral operators P and Q then,

$P^2 = \frac{1}{(2\pi i)^2}\int\limits_{\acute{\gamma}}\int\limits_\gamma R_\mu^L(M)R_\mu^L(M)d\mu d\lambda =$

$$= \frac{1}{(2\pi i)^2}\left(\int\limits_{\acute{\gamma}}\frac{d\lambda}{\lambda - \mu}\int\limits_\gamma R_\mu^L(M)d\mu + \int\limits_{\acute{\gamma}} R_\lambda^L(M)d\lambda\int\limits_\gamma\frac{d\mu}{\mu - \lambda}\right) = P,$$

according to the Fubini theorem, residue theorems and the equation (2). We prove Q as a projector operator Similarly. •

We observe that $\forall\ u \in \mathfrak{U}$,

$$LPu = QLu. \tag{3}$$

$$MPu = QMu. \tag{4}$$

Let $\mathfrak{U}^0$ ($\mathfrak{U}^1$) = ker$P$ (im$P$), $\mathfrak{F}^0$ ($\mathfrak{F}^1$) = ker$Q$ (im$Q$), and L_k (M_k) is the restriction of an operator L (M) to $\mathfrak{U}^k$, $k = 0{,}1$.It follows from the lemma 1 that the projectors P and Q split the spaces $\mathfrak{U}$ and $\mathfrak{F}$ into direct sums $\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1$ and $\mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^1$.

Theorem 3 .(Sviridyuk-Jawad Al-Delfi Theorem) Let an operator $M(L{,}\sigma)$-bounded, then
(i) operators $L_k{,}M_k \in \mathcal{L}(\mathfrak{U}^k;\mathfrak{F}^k)$, $k = 0{,}1$;
(ii) there are operators $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1;\mathfrak{U}^1)$ and $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0;\mathfrak{U}^0)$.

Proof. Clearly, the statement(i) follows from the relations (3),(4).

(ii) Using the equation (1) when $\lambda = 0$, by the continuity of an operator M, and by Lemma 1, let $f^o \in \mathfrak{F}^0$, then

$$M\frac{1}{2\pi i}\int\limits_\gamma(\mu L - M)^{-1}f^o\frac{d\mu}{\mu} = -\frac{1}{2\pi i}\int\limits_\gamma\frac{d\mu}{\mu}f^o + \frac{1}{2\pi i}\int\limits_\gamma L_\mu^L(M)f^o d\mu = -f^o.$$

Now, let $u^o \in \mathfrak{U}^0$, then

$$\frac{1}{2\pi i}\int\limits_\gamma(\mu L - M)^{-1}\frac{d\mu}{\mu}Mu^o = -\frac{1}{2\pi i}\int\limits_\gamma\frac{d\mu}{\mu}u^o + \frac{1}{2\pi i}\int\limits_\gamma R_\mu^L(M)u^o d\mu = -u^o.$$

This means that an operator M_0^{-1} is equal to a restriction of an operator $-\frac{1}{2\pi i}\int\limits_\gamma(\mu L - M)^{-1}\frac{d\mu}{\mu}$ on the subspace $\mathfrak{F}^0$. Also, by Lemma 1, an operator L_1^{-1} is equal to a restriction of an operator $\frac{1}{2\pi i}\int\limits_\gamma(\mu L - M)^{-1}d\mu$. on a subspace $\mathfrak{F}^1$. •

According to Theorem 3, there are operators: $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{U}^0)$, $S = L_1^{-1}M_1 \in \mathcal{L}(\mathfrak{U}^1)$.

We say that ∞ is a *removable singularity point* if $H \equiv \mathbb{O}$; *pole of order* $p \in \mathbb{N}$ if $H^p \neq \mathbb{O}$ and $H^{p+1} \equiv \mathbb{O}$; *essential singularity point* of L-resolvent of an operator M if $H^k \neq \mathbb{O}$, $k \in \mathbb{N}$of L-resolvent of an operator M. An operator $M(L{,}\sigma)$-*bounded* is called $M(L{,}p)$-*bounded* if ∞ – is a nonessential singularity point of its L-resolvent.

Cauchy Condition for Nonhomogeneous Equations

Let operators $L{,}M \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, where $\mathfrak{U}$ and $\mathfrak{F}$ are quasi-Banach spaces. Consider linear Sobolev type equation

$$L\dot{u} = Mu + f. \tag{5}$$

A vector function $u : (a{,}b) \to \mathfrak{U}$,$(a{,}b) \subset \mathbb{R}$ satisfying equation (5) is called a *a solution* of this equation. The solution $u = u(t)$ of the equation (5) will be called *the solution of a Cauchy problem:*

$$u(0) = u_0. \tag{6}$$

For equation (5) (briefly, a solution to the problem (5), (6)) it in addition satisfies the Cauchy condition (6) for some $u_0 \in \mathfrak{U}$.

A mapping $U^\bullet \in C^\infty(\mathbb{R};\mathcal{L}(\mathfrak{U}))$ is called a group of solving operators of a homogeneous equation (5) if $U^sU^t = U^{s+t}$ $\forall s{,}t \in \mathbb{R}$, and for any $u_0 \in \mathfrak{U}$, the vector function $u(t) = U^tu_0$ has solutions of a homogeneous equation (5).

Theorem 4. Let the operator M be (L,p)-bounded , $p \in \{0\} \cup \mathbb{N}$, point $0 \in [a,b]$. Then for any $f \in C^{\infty}([a,b];\mathfrak{F})$ and any

$$u_0 \in \left\{ u \in \mathfrak{U} : u = -\sum_{k=0}^{\infty} H^k M_0^{-1}(\mathbb{I} - Q) f^{(k)}(0) \right\},$$

there is a unique solution $u \in C^{\infty}([a,b];\mathfrak{U})$ of problem (5) , (6) , which also has the form

$$u(t) = -\sum_{k=0}^{p} H^k M_0^{-1}(\mathbb{I} - Q) f^{(k)}(t) + U^t u_0 + \int_0^t U^{t-s} L_1^{-1} Q f(s) ds.$$

Here the family of operators $\{U^t : t \in \mathbb{R}\}$ is a solving group of a homogeneous equation (5) .

Proof. According to Theorem 3, problem (5), (6) is reduced to two problems:

$$H\dot{u}^0 = u^0 + M_0^{-1} f^0, u^0(0) = u_0^0 \tag{7}$$

$$\dot{u}^1 = Su^1 + L_1^{-1} f^1, u^1(0) = u_1^0, \tag{8}$$

where $u^m vectors \in \mathfrak{U}^m$, $f^m \in \mathfrak{F}^m$, m = 0.1. Operator $S \in \mathcal{L}(\mathfrak{U}^1)$, so problem (8) has a unique solution $u^1 \in C^{\infty}([a,b];\mathfrak{U})$, is presented as:

$$u^1(t) = e^{tS} u_0^1 + \int_0^t e^{(t-s)S} L_1^{-1} Q f^1(s) ds, \text{ where } t \in [a,b].$$

In order to consider the problem (7), we assume additionally that ∞ is a nonessential singularity point of L-resolvent of an operator M.

Then, by successive differentiatinn equation (7) with respect to t and premultiplying it by the operator H from the left, we finally get

$$u^0(t) = -\sum_{k=0}^{p} H^k M_0^{-1} \overset{0}{f}{}^{(k)}(t), t \in [a,b].$$

Hence, it is clear that problem (7) is unsolvable if

$$u_0^0 \neq -\sum_{k=0}^{p} H^k M_0^{-1} \overset{0}{f}{}^{(k)}(0). \tag{9}$$

On the other hand, if (9) does not hold, then problem (7) has a unique solution $u^0 \in C^{\infty}([a,b];\mathfrak{U}^0)$. Let us describe the set of admissible intinal values of problem (7), i.e. such that problem (7) is uniquely solvable. According to (9) and Theorem 3, this set has the form

$$\mathfrak{P}_f = \{u \in \mathfrak{U} : (\mathbb{I} - Q)(Mu + \sum_{k=0}^{p} L_0 M_0^{-1} f^{(k)}(0)) = 0\}.$$

Therefore, $u_0 \in \left\{ u \in \mathfrak{U} : u = -\sum_{k=0}^{\infty} H^k M_0^{-1}(\mathbb{I} - Q) f^{(k)}(0) \right\}$. Thus, the theorem is proved.•

Example . Let $\mathfrak{U} = l_q^{m+2}$, $\mathfrak{F} = l_q^m$, $L,M \in \mathcal{L}(l_q^{m+2}; l_q^m)$ set by the formulas $L = \lambda - \Lambda$, $M = \alpha\Lambda$, where $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{R}\backslash\{0\}$. Since $\sigma^L(M) = \left\{ \mu_k \in \mathbb{C} : \mu_k = \dfrac{\alpha\lambda_k}{\lambda - \lambda_k},\ k \in \mathbb{N}\backslash\{l : \lambda = \lambda_l\} \right\}$, then $M(L,\sigma)$-*bounded* .Also,

$$\mathfrak{U}^0 = \begin{cases} \{\mathbf{0}\}, & \text{if } \lambda \notin \{\lambda_k\}; \\ \{u \in \mathfrak{U} : \ u_k = 0, \, k \in \mathbb{N}\backslash\{l : \, \lambda = \lambda_l\}\} \, ; \end{cases}$$

$$\mathfrak{U}^1 = \begin{cases} \mathfrak{U}, & \text{if } \lambda \notin \{\lambda_k\}; \\ \{u \in \mathfrak{U} : \ u_l = 0, \, \lambda_l = \lambda\} \, . \end{cases}$$

The subspaces $\mathfrak{F}^k$, $k = 0{,}1$ are defined similarly.We observe that an operator $M(L{,}0)$-bounded. Indeed, it is easy to show that in this case the operator $H = \mathbb{O}$.

Consider the Barenblatt-Zheltov-Kochina equation:

$$(\lambda - \Lambda)\dot{u} = \alpha\Lambda u + f. \tag{10}$$

If take Cauchy condition with equation (10), we have problem (10), (6).

Corollary. Let $0 \in [a{,}b]$, then for any $\lambda \in \mathbb{R}$, $\alpha \in \mathbb{R}\backslash\{0\}$, $q \in \mathbb{R}_+$, $f \in C^\infty([a{,}b]; \ell_q^m)$ and any

$$u_0 \in \{u \in \ell_q^{m+2} : u = -M_0^{-1}(\mathbb{I} - Q)f(0)\},$$

there is a unique solution $u \in C^\infty([a{,}b]; \ell_q^{m+2})$ of the problem (10), (6), which also has the form:

$$u(t) = -M_0^{-1}(\mathbb{I} - Q)f(t) + U^t u_0 + \int\limits_0^t U^{t-s} L_1^{-1} Q f(s) ds.$$

Proof. Since an operator $M(L{,}0)$-bounded, which is shown in the previous example, where $H = \mathbb{O}$ and according to Theorem 4, then the desired result is satisfied.•

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