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ON RELATIVELY σ-BOUNDED OPERATORS IN QUASI-BANACH SPACES

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Abstract: Relatively σ-bounded operators are introduced and studied in quasi-Banach spaces. Abstract results are illustrated by examples from quasi-Sobolev spaces and Laplace' Quasi-operator.

Key words and phrases: Splitting Theorem, Quasi-Banach spaces, Quasi-Sobolev spaces, Laplace' Quasi-operator.

ОБ ОТНОСИТЕЛЬНО ОГРАНИЧЕННЫХ $\sigma-$ ОПЕРАТОРАХ В КВАЗИБАНАХОВЫХ ПРОСТРАНСТВАХ

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Аннотация: Вводятся и изучаются относительно σ-ограниченные операторы в квазибанаховых пространствах. Абстрактные результаты иллюстрированы примерами из квазисоболевых пространств и квазиоператора Лапласа.

Ключевые слова: теорема о расщеплении, Квазибанаховы пространства, Квазисоболевы пространства, Квазиоператор Лапласа.

INTRODUCTION

The theory of σ-bounded operators in Banach spaces has studied, and has numerous applications and even extended to locally convex spaces ([1]–[5]).

A set of all monotonically increasing eigen values $\{\lambda_k\} \subset \mathbf{R}_+$, $k \in \mathbf{N}$ such that $\lim\limits_{k\to\infty} \lambda_k = +\infty$ of a Laplace operator, was used to construct quasi-Sobolev spaces l_p^m, $p \in (0, +\infty)$,$m \in \mathbb{R}$: $l_p^m = \left\{ u = \{u_k\} : \sum\limits_{k=1}^{\infty} \left(\lambda_k^{\frac{m}{2}} |u_k| \right)^p < +\infty \right\}$, and then to define quasi- Laplace operator $\Lambda u = \lambda_k u_k$ on these spaces [7, 8].

In this article, we introduce concept of quasi-Banach space [6, 8] and extend the theory of σ-bounded operators to this concept, and give some results with examples on quasi-Sobolev spaces and quasi- Laplac operators.

1. QUASI-BANACH SPACE

Quasi-normed space $(\mathfrak{U}, {}_{\mathfrak{U}}\|\cdot\|)$ (simply $\mathfrak{U}$)is a vector space $\mathfrak{U}$ over a field $\mathbb{F}$ with quasi-norm ${}_q\|\cdot\|$, which differs from a norm $\|\cdot\|$only by «triangle inequality»: $\forall u,v \in \mathfrak{U}$, ${}_{\mathfrak{U}}\|u+v\| \leqslant c({}_{\mathfrak{U}}\|u\| + {}_{\mathfrak{U}}\|v\|)$, where $C \geqslant 1$. If a constant $c = 1$, then ${}_q\|\cdot\| = \|\cdot\|$.

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A sequence $\{x_k\} \subset \mathfrak{U}$ is called *convergent* to $x \in \mathfrak{U}$ if $\lim\limits_{k\to\infty} x_k = x$. A sequence is called *fundamental* if $\lim\limits_{k,r\to\infty}(x_k - x_r) = 0$. A space $\mathfrak{U}$ is called *quasi-Banach* if any fundamental sequence in $\mathfrak{U}$ converges to some point in it.

We say that, a quasi-Banach space $\mathfrak{U}$ is *density and continuously embedded in* a quasi-Banach space$\mathfrak{F}$ $(\mathfrak{U} \hookrightarrow \mathfrak{F})$ if $\mathfrak{U} \subset \mathfrak{F}$; closure $\overline{\mathfrak{U}} = \mathfrak{F}$; and for all $u \in \mathfrak{U}$ ${}_q\|u\|_{\mathfrak{U}} \geqslant M_q\|u\|_{\mathfrak{F}}$, where $M \in \mathbb{R}_+$.

Let $\mathfrak{U}$ and $\mathfrak{F}$ — - quasi-Banach spaces, a linear operator $L : \mathfrak{U} \to \mathfrak{F}$ is called bounded if ${}_q\|Lu\|_{\mathfrak{F}} \leqslant M_q\|u\|_{\mathfrak{U}}$, for all $u \in \mathfrak{U}$, $M \in \mathbb{R}_+$, and is continuous if $\lim\limits_{k\to\infty} Lu_k = L\left(\lim\limits_{k\to\infty} u_k\right)$, for every convergent sequence $\{ u_k\} \subset \mathfrak{U}$. A linear operator L is called toplinear isomorphism if L and its inverse L^{-1} are continuous.

$\mathcal{L}(\mathfrak{U};\mathfrak{F})(\mathcal{L}(\mathfrak{U})$ $(\equiv \mathcal{L}(\mathfrak{U};\mathfrak{U}))$ — set of all linear continuous operators is quasi-Banach space with the quasi-norm: ${}_{\mathcal{L}(\mathfrak{U};\mathfrak{F})}\|L\| = \sup\limits_{\mathfrak{U}\|u\|=1} {}_{\mathfrak{F}}\|Lu\|$.

Example 1. For every $p \in (0, +\infty)$, $m \in \mathbb{R}$, l_p^m be a quasi-Banach space with ${}_q\|u\|_m = \left(\sum\limits_{k=1}^{\infty} \lambda_k^{mp/2}|u_k|^p\right)^{1/p}$, and it is a Banach space when $1 < p < \infty$, where c = 1. For every $m \geqslant n$, we have $l_p^m \hookrightarrow l_p^n$, and note that a constant c $= 2^{1/p}$ when $p \in (0,1)$. Also, a quasi-Laplace operator $\Lambda : l_p^{m+2} \to l_p^m, \Lambda u = \{\lambda_k u_k\}$ has a continuous inverse $\Lambda^{-1} \in \mathcal{L}(l_q^m; l_q^{m+2}), \Lambda^{-1}u = \{\lambda_k^{-1}u_k\}$-*quasi Green's operator*, then Λ is toplinear isomorphism operator.

2. RELATIVELY σ-BOUNDED OPERATORS

Let $\mathfrak{U}$ и $\mathfrak{F}$ — quasi-Banach spaces, operators $L,M \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, we introduce the L-resolvent set $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})\}$ and L-spectrum $\sigma^L(M) = \mathbb{C}\backslash\rho^L(M)$ of an operator M. Suppose $\rho^L(M) \neq \varphi$, then operator-functions:

$(\mu L - M)^{-1}, R_\mu^L(M) = (\mu L - M)^{-1}L$ and $L_\mu^L(M) = L(\mu L - M)^{-1}$ are called L resolvent, right and left L resolvent of an operator M respectively. We observe that :

$$LR_\mu^L(M) = L_\mu^L(M)L. \tag{1}$$

$$MR_\mu^L(M) = L_\mu^L(M)M. \tag{2}$$

Remark 1. Since $(\lambda L - M) = (\mu L - M) + (\lambda - \mu)L$, $\forall\mu, \lambda \in \rho^L(M)$, then we have:

$$(\mu L - M)^{-1}(\lambda L - M)^{-1} = \mathbb{I} + (\lambda - \mu)R_\lambda^L(M) \tag{3}$$

$$R_\lambda^L(M) - R_\mu^L(M) = (\mu - \lambda)R_\mu^L(M)R_\lambda^L(M), \tag{4}$$

$$L_\lambda^L(M) - L_\mu^L(M) = (\mu - \lambda)L_\mu^L(M)L_\lambda^L(M). \tag{5}$$

An operator M is said to be *spectrally bounded* with respect to an operator L (shortly, $M(L,\sigma)$-*bounded*) if

$$\exists a \in \mathbb{R}_+ \ \forall\mu \in \mathbb{C} \ (|\mu| > a) \ \Rightarrow \ \left(\mu \in \rho^L(M)\right).$$

Not all operators are relatively σ-bounded,as shown in the following:

Example 2. Let $\mathfrak{U} = l_p^{m+2}$, $\mathfrak{F} = l_p^m$, $L,M \in \mathcal{L}(l_p^{m+2}; l_p^m)$ set by the formulas $L = \lambda - \Lambda$, $M = \alpha\Lambda$, where $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{R}\backslash\{0\}$. Since $\sigma^L(M) = \left\{\mu_k \in \mathbb{C} : \mu_k = \dfrac{\alpha\lambda_k}{\lambda - \lambda_k}, k \in \mathbb{N}\backslash\{l : \lambda = \lambda_l\}\right\}$, then $M(L,\sigma)$-*bounded* .

Example 3. Consider Green's quasi-operator $\Lambda^{-1} \in \mathcal{L}(l_p^m; l_p^{m+2})$ as $\Lambda^{-1} \in \mathcal{L}(l_p^m; l_p^m)$. Let $\mathfrak{U} = \mathfrak{F} = l_p^m$, operators $L = \Lambda^{-1}$, $M = \mathbb{I}$ be an identity operator on l_p^m. Since $\sigma^L(M)$ consists of the points $\{\lambda_k\}$, then $M(L,\sigma)$ is not bounded.

Let an operator $M(L,\sigma)$-bounded, and a contour $\gamma = \{\mu \in \mathbb{C} : \ |\mu| = r > a\}$. Consider integrals of the type F. Rissa

$$P = \frac{1}{2\pi i}\int\limits_{\gamma} R^L_\mu(M)d\mu, \qquad Q = \frac{1}{2\pi i}\int\limits_{\gamma} L^L_\mu(M)d\mu. \tag{6}$$

Lemma 1. Let an operator $M(L,\sigma)$-bounded, then operators $P \in \mathcal{L}(\mathfrak{U})$ and $Q \in \mathcal{L}(\mathfrak{F})$ are projectors.

Proof. Take a contour $\acute{\gamma} = \{\lambda \in \mathbb{C} : \ |\lambda| = \acute{r} > r\}$. According to the analyticity of integral operators P and Q then,
$P^2 = \frac{1}{(2\pi i)^2}\int\limits_{\acute{\gamma}}\int\limits_{\gamma} R^L_\mu(M)R^L_\mu(M)d\mu d\lambda =$

$$= \frac{1}{(2\pi i)^2}\left(\int\limits_{\acute{\gamma}}\frac{d\lambda}{\lambda-\mu}\int\limits_{\gamma} R^L_\mu(M)d\mu + \int\limits_{\acute{\gamma}} R^L_\lambda(M)d\lambda\int\limits_{\gamma}\frac{d\mu}{\mu-\lambda}\right) =$$

$$= \frac{1}{2\pi i}\int\limits_{\gamma} R^L_\mu(M)d\mu = P,$$

according to the Fubini theorem, residue theorems and the equation (4). Similarly, we prove Q is projector by help the equation(5). •

Remark 2. According to equalities (1) and (2), $\forall\ u \in \mathfrak{U}$, we have the following relations:

$$LPu = QLu. \tag{7}$$

$$MPu = QMu. \tag{8}$$

Let $\mathfrak{U}^0$ ($\mathfrak{U}^1$) = ker$P$ (im$P$), $\mathfrak{F}^0$ ($\mathfrak{F}^1$) = ker$Q$ (im$Q$), and L_k (M_k) is the restriction of an operator L (M) to $\mathfrak{U}^k$, $k = 0,1$.It follows from the lemma 1 that the projectors P and Q split the spaces $\mathfrak{U}$ and $\mathfrak{F}$ into direct sums $\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1$ and $\mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^1$.

Example 4. Let $\mathfrak{U}$, $\mathfrak{F}$ and L,M are same as in Example 2,then
$\mathfrak{U}^0 = \begin{cases} \{\mathbf{0}\}, & \text{if } \lambda \notin \{\lambda_k\}; \\ \{u \in \mathfrak{U} : \ u_k = 0,\ k \in \mathbb{N}\backslash\{l : \ \lambda = \lambda_l\}\}; \end{cases}$
$\mathfrak{U}^1 = \begin{cases} \mathfrak{U}, & \text{if } \lambda \notin \{\lambda_k\}; \\ \{u \in \mathfrak{U} : \ u_l = 0,\ \lambda_l = \lambda\}. \end{cases}$

$$Pu = \begin{cases} u = \{u_k\}, & \text{if } \lambda \notin \{\lambda_k\}; \\ \{(u_k : k \in \mathbb{N}\backslash\{l : \lambda_l = \lambda\}) & \text{and} \quad (u_l = 0 : \lambda_l = \lambda)\} \end{cases}$$

The subspaces $\mathfrak{F}^k$, $k = 0,1$ and Qu are defined similarly.

We introduce Sviridyuk-Jawad Al-Delfi Theorem-splitting Theorem in quasi-Banach space.

Theorem 1. (Sviridyuk-Jawad Al-Delfi Theorem) Let an operator $M(L,\sigma)$-bounded, then
(i) operators $L_k, M_k \in \mathcal{L}(\mathfrak{U}^k; \mathfrak{F}^k)$, $k = 0,1$;
(ii) there are operators $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1; \mathfrak{U}^1)$ and $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0; \mathfrak{U}^0)$.

Proof. Clearly, the statement(i) follows from the relations (7),(8).

(ii) Using the equation (3) when $\lambda = 0$, by the continuity of an operator M, and by Lemma 1, let $f^o \in \mathfrak{F}^0$, then

$$M\frac{1}{2\pi i}\int\limits_{\gamma}(\mu L - M)^{-1}f^o\frac{d\mu}{\mu} = -\frac{1}{2\pi i}\int\limits_{\gamma}\frac{d\mu}{\mu}f^o + \frac{1}{2\pi i}\int\limits_{\gamma} L^L_\mu(M)f^o d\mu = -f^o.$$

Now, let $u^o \in \mathfrak{U}^0$, then

$$\frac{1}{2\pi i}\int\limits_{\gamma}(\mu L - M)^{-1}\frac{d\mu}{\mu}Mu^o = -\frac{1}{2\pi i}\int\limits_{\gamma}\frac{d\mu}{\mu}u^o + \frac{1}{2\pi i}\int\limits_{\gamma}R^L_\mu(M)u^o d\mu = -u^o.$$

This means that an operator M_0^{-1} is equal to a restriction of an operator $-\frac{1}{2\pi i}\int\limits_{\gamma}(\mu L - M)^{-1}\frac{d\mu}{\mu}$ on the subspace $\mathfrak{F}^0$. Also, by Lemma 1, an operator L_1^{-1} is equal to a restriction of an operator $\frac{1}{2\pi i}\int\limits_{\gamma}(\mu L - M)^{-1}d\mu$. on a subspace $\mathfrak{F}^1$. •

According to the previous theorem, there are operators:
$H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{U}^0)$, $S = L_1^{-1}M_1 \in \mathcal{L}(\mathfrak{U}^1)$.

Corollary 1. Let the conditions of Theorem 1 be satisfied, then for all $\mu \in \mathbb{C}$: $|\mu| > a$ we have:

$$(\mu L - M)^{-1} = -\sum_{k=0}^{\infty}\mu^k H^k M_0^{-1}(\mathbb{I} - Q) + \sum_{k=1}^{\infty}\mu^{-k}S^{k-1}L_1^{-1}Q. \tag{9}$$

Proof. Operator-function $(\mu L_0 - M_0)^{-1}$ is an entire function by Theorem 1. Therefore, we can represent this function by a Taylor series: $(\mu L_0 - M_0)^{-1} = (\mu H - \mathbb{I})^{-1}M_0^{-1} = (-\sum\limits_{k=0}^{\infty}\mu^k H^k)M_0^{-1}$, is absolute and uniformly convergent on any compact set in $\mathbb{C}$. For $(\mu L_1 - M_1)^{-1}$ we have $(\mu L_1 - M_1)^{-1} =$

$$= (\mu\mathbb{I} - S)^{-1}L_1^{-1} = \ mu^{-1}(\mathbb{I} - \mu^{-1}S)^{-1}L_1^{-1} = \mu^{-1}(\sum_{k=0}^{\infty}\mu^{-k}S^k)L_1^{-1}.$$

Hence, for $M(L,\sigma)$-bounded by virtue of Theorem 1 and the last two expansions, we obtain equation (9).•

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