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ON QUASI-SOBOLEV SPACES

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Abstract: The notion quasi-Sobolev spaces is introduced in the article based on the concept quasi-norms. Completeness of these spaces can be proved on the appropriate quasi-norms and continuous embedding of these spaces is shown in the work. Also concepts quasi-operators Laplace and Green are introduced and shown that these quasi-operators are toplinear isomorphisms.

Key words and phrases: quasi-norm, quasi-Banach Space, quasi-Sobolev spaces, Laplas' quasi-operator, Grins' quasi-operator.

ОБ КВАЗИСОБОЛЕВЫ ПРОСТРАНСТВА

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Аннотация: На основе понятия квазинормы в статье вводится понятие квазисоболевских пространств. Показывается их полнота относительно соответствующих квазинорм и непрерывность вложений этих пространств.Также вводятся понятия квазиоператоров Лапласа и Грина и показывается,что эти квазиоператоры являются топлинейными изоморфизмами.

Ключевые слова: квазинормы, квазибанахово пространство, квазисоболевы пространства, квазиоператор Лапласа, квазиоператор Грина.

INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a class boundaryC^{∞}, and $W^{m}_{p}(\Omega)$, $1 \leqslant p < \infty$, $m \in \mathbf{N} \cup \{0\}]$ – Sobolev space, where $W^{\circ}_{p}(\Omega) = L_p(\Omega)$ is a Lebesgue space. The Sobolev embedding theorem is also well known : for all $0 < m \leqslant l < \infty$,$1 \leqslant p \leqslant q < \infty$ such that $\frac{1}{p} - \frac{m-l}{n} \leqslant \frac{1}{q} < 1$ then $W^{m}_{p}(\Omega)$ is dense and continuous(even compact) embedded in $W^{l}_{p}(\Omega)$ [1], that is:

$$W^{m}_{p}(\Omega) \hookrightarrow W^{l}_{p}(\Omega) \tag{1}$$

Also well known, Laplace operator $-\Delta$, which is defined by the form :

$$-\langle \Delta u, v \rangle = \sum_{m=1}^{n} \int_{\Omega} u_{x_m} v_{x_m} dx,$$

sets a toplinear isomorphism operator([2], sec. 3):

$$-\Delta : \overset{\circ}{W}{}^{1}_{2}(\Omega) \to W^{-1}_{2}(\Omega), \tag{2}$$

such that:
$$\overset{\circ}{W}{}_2^1(\Omega) \hookrightarrow L_2(\Omega) \hookrightarrow W_2^{-1}(\Omega), \qquad (3)$$
where $W_2^{-1}(\Omega)$ is dual space of a Sobolev space $\overset{\circ}{W}{}_2^1(\Omega)$

Furthermore, Let $\{\lambda_k\} \subset \mathbb{R}_+$ — set of eigenvalues of a Laplace operator $-\Delta$ which is monotonically increasing sequence such that $\lim\limits_{k\to\infty} \lambda_k = +\infty$. We construct

$$l_2^1 = \left\{ x = \{x^k\} : \sum_{k=1}^{\infty} \lambda_k |x^k|^2 < +\infty \right\},$$

$$l_2^{-1} = \left\{ x = \{x^k\} : \sum_{k=1}^{\infty} \lambda_k^{-1} |x^k|^2 < +\infty \right\},$$

and observe toplinear isomorphism operators: $l_2^1 \cong \overset{\circ}{W}{}_2^1(\Omega)$, $l_2^{-1} \cong W_2^{-1}(\Omega)$, and also dense and continuous embeddings:
$$l_2^1 \hookrightarrow l_2 \hookrightarrow l_2^{-1}, \qquad (4)$$
which is coming from (2). We observe that l_2^1, l_2^{-1} are Banach spaces with norms $\|x\|_1^2 = \sum\limits_{k=1}^{\infty} \lambda_k |x^k|^2$ and $\|y\|_{-1}^2 = \sum\limits_{k=1}^{\infty} \lambda_k^{-1} |y^k|^2$ consequently. We introduce a quasi-operator Laplace:
$$\Lambda x = \lambda_k x^k. \qquad (5)$$
Since $\|\Lambda x\|_{-1} = \|x\|_1$, then from (5) and according to (2), (4) it is easy to obtain a toplinear isomorphism operator $\Lambda : l_2^1 \to l_2^{-1}$. The inverse of Λ is a quasi-operator Green Λ^{-1} that is defined as:
$$\Lambda^{-1} y = \lambda_k^{-1} y^k. \qquad (6)$$

The article is devoted to the transfer of the ideology described above to the sequence space l_p, $p \in (0,\infty)$ with extension of (1) to construct sequence spaces of power $m \in \mathbb{R}$ which have called quasi-Sobolev spaces and are defined as:

$$l_p^m = \left\{ x = \{x^k\} : \sum_{k=1}^{\infty} \lambda_k^{mp/2} |x^k|^p < +\infty \right\},$$

where,$\{\lambda_k\}$ is monotonically increasing sequence of positive numbers such that $\lim\limits_{k\to\infty} \lambda_k = +\infty$. When $m = 0$, $l_p^0 = l_p$

The article contains three sections, the first section contains the basic facts of the concept of quasi-Banach spaces, and in the second section, analog of the Sobolev embedding theorem is presented. In third section, the Laplace quasi-operator is introduced and is proved as toplinear isomorphism.

3. QUASI-SOBOLEV SPACE

Let $\mathfrak{U}$ — real vector space.

Definition . A function ${}_q\|\cdot\| : \mathfrak{U} \to R$ is called a quasi-norm if it is satisfied the following properties:

(i) $\forall u \in \mathfrak{U}$, ${}_q\|u\| \geqslant 0$, such that ${}_q\|u\| = 0 \Leftrightarrow u = \mathbf{0}$;

(ii) $\forall u \in \mathfrak{U}\ \ \forall \alpha \in \mathbb{R}\ \ {}_q\|\alpha u\| = |\alpha|_q \|u\|$;

(iii) $\forall u,v \in \mathfrak{U}\ \ {}_q\|u + v\| \leqslant c({}_q\|u\| +_q \|v\|)$, where $c \in [1, +\infty)$.

A quasi-normed space is$(\mathfrak{U},_q\|\cdot\|)$ or simply $\mathfrak{U}$.

A sequence $\{x_k\}\subset\mathfrak{U}$ is called *convergent* to $x\in\mathfrak{U}$ if $\lim\limits_{k\to\infty}{}_q\|x_k-x\|=0$, or this fact writes as: $\lim\limits_{k\to\infty}x_k=x$. A sequence is called *fundamental* if $\lim\limits_{k,r\to\infty}(x_k-x_r)=0$.

A space $\mathfrak{U}$ is called *quasi-Banach* if any fundamental sequence in this space converges to some point in it. We immediately note that any Banach space is a quasi-Banach space, and the opposite is not true in generally.

Example. Sequence spaces l_p be quasi-Banach spaces when $p\in(0,+\infty]$, while they are Banach spaces only when $p\in[1,+\infty]$.

Theorem 1.*For every $p\in(0,+\infty)$, $m\in\mathbb{R}$, a space l_p^m be a quasi-Banach space with a function:* ${}_q\|x\|_m=\left(\sum\limits_{k=1}^{\infty}\lambda_k^{mp/2}|x^k|^p\right)^{1/p}$.

P r o o f this fact analogues section. 4.2 [3]. We also note that a constant c $=2^{1/p}$ when $p\in(0,1)$ and c $=1$ when $p\in[1,+\infty)$. •

4. THE EMBEDDING THEOREM

Let $\mathfrak{U}$ and $\mathfrak{F}$ — two quasi-Banach spaces. We say that:

— $\mathfrak{U}$ *embedded in* $\mathfrak{F}$, if $\mathfrak{U}$ subset of $\mathfrak{F}$, i.e. $\mathfrak{U}\subset\mathfrak{F}$;

— $\mathfrak{U}$ *dense embedded in* $\mathfrak{F}$, if moreover closure $\overline{\mathfrak{U}}=\mathfrak{F}$;

— $\mathfrak{U}$ *dense and continuous embedded in* $\mathfrak{F}$, if moreover for all $u\in\mathfrak{U}$ ${}_q\|u\|_{\mathfrak{U}}\geqslant M_q\|u\|_{\mathfrak{F}}$, where $M\in\mathbb{R}_+$– a constant independent of u.

Theorem 2. *For every $p\in(0,+\infty]$, $m\in\mathbb{R}$, $l\leqslant m$ then l_p^m is dense and continuous embedded in l_p^l, that is, $l_p^m\hookrightarrow l_p^l$.*

proof. $l_p^m\subset l_p^l$ is obvious. We prove dense embedded l_p^m in l_p^l. Let $x\in l_p^l$, and we consider sequence $\{x_k\}$, where

$$x_1=(x^1,0,0,\ldots),\ x_2=(x^1,x^2,0,0,\ldots),\ \ldots,\ x_k=(x^1,x^2,\ldots,x^k,0,0,\ldots).$$

It is obvious, $\{x_k\}\subset l_p^m$, such that $\lim\limits_{k\to\infty}x_k=x$ in a quasi-norm of l_p^l. Continuous embedded $l_p^m\hookrightarrow l_p^l$ is obvious.•

5. 3. QUASI-OPERATOR LAPLAC

Let $\mathfrak{U}$ and $\mathfrak{F}$ — quasi-Banach spaces, a linear operator $S:\mathfrak{U}\to\mathfrak{F}$ is called continuous if $domS=\mathfrak{U}$ and ${}_q\|u\|_{\mathfrak{U}}\geqslant M_q\|Su\|_{\mathfrak{F}}$, for all $u\in\mathfrak{U}$, $M\in\mathbb{R}_+$ — a constant independent of u. A continuous linear operator S is called toplinear isomorphism if there exists an inverse operator $S^{-1}:\mathfrak{F}\to\mathfrak{U}$, which is also continuous.

We define a quasi-Laplace operator $\Lambda x=\lambda_k x^k$, where $x\in l_p^m$ by formula (5).

Theorem 3.*For all $p\in(0,+\infty)$, a quasi-Laplace operator $\Lambda:l_p^{m+2}\to l_p^m$ — toplinear isomorphism.*

proof. It is clear that Λ is continuous –

$${}_q\|\Lambda x\|_m=\left(\sum_{k=1}^{\infty}\lambda_k^{(m/2)+1}|x^k|^p\right)^{1/p}={}_q\|x\|_{m+2}.$$

We construct quasi-Green operator $\Lambda^{-1}x=\lambda_k^{-1}x$,where $x\in l_p^{m+2}$ by formula (6) Obviously, $\Lambda\Lambda^{-1}x=x$ for all $x\in l_p^m$, and $\Lambda^{-1}\Lambda x=x$ for all $x\in l_p^{m+2}$. Moreover, Λ^{-1} is continuous –

$$ {}_q\|\Lambda^{-1}x\|_{m+2} = \left(\sum_{k=1}^{\infty} \lambda_k^{(m/2)-1} |x^k|^p \right)^{1/p} = {}_q\|x\|_m. \bullet $$

Remark . An extension of the results of this article to the case of complex spaces l_p, $p \in (0, +\infty)$, is obvious.

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