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SYSTEMS WITH DIODE NONLINEARITY IN SOME ELECTRICAL CIRCUITS' MODELING

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Abstract: This paper is concerned with so-called *systems with diode nonlinearities*. We discuss such systems first from the mathemical viewpoint and recall some facts from the theory of cones, projections, and differential inclusions. Afterwards we show that certain electrical circuits with diode converters may be viewed as natural examples. In particular, our approach provides a universal and "automatic" description for all circuits which satisfy a so-called *LC-condition*. This condition is either fulfilled for a circuit, or may be achieved by adding small inductivities and capacitiesto a specific part of the circuit.

Key words and phrases: Normal cone, tangent cone, adjoint cone, diode nonlinearity, differential inclusion, electrical circuit, diode converter, LC-condition.

СИСТЕМЫ С ДИОДНОЙ НЕЛИНЕЙНОСТЬЮ В МОДЕЛИРОВАНИИ НЕКОТОРЫХ ЭЛЕКТРИЧЕСКИХ ЦЕПЕЙ

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Аннотация: Эта статья посвящается так называемым системам с диодными нелинейностями. Обсуждаются такие системы сначала с математической точки зрения, повторяя некоторые результаты из теории конусов, проекций и дифференциальных включений. Затем показывается, как некоторые электрические цепи с диодными преобразователями могут служить естественными примерами. В частности, наш подход дает универсальное и "автоматическое" описание всех цепей, которые удовлетворяют так называемому LC-условию. Это условие либо выполняется для цепи, либо его можно получить, включая в определенные места цепи малые индуктивности или ёмкости.

Ключевые слова: нормальный конус, касательный конус, сопряженный конус, диодная нелинейность, дифференциальное включение, электрическая цепь, диодный преобразователь, LC-условие.

In the theory of electrical circuits there exist various descriptions, both as algebraic and differential systems, but only in the linear case. A great advantage is here the possibility of programming this description of a linear circuit which provides a universal method. On the other hand, for electrical circuits with nonlinear elements no such approach exists and, even worse, probably cannot exist at all. In the literature a certain variety of methods may be found also for nonlinear circuits, but each method applies only to one spedific type of circuits individually.

The aim of this paper is to provide a universal and "automatic" description for all circuits which satisfy a so-called LC-condition. This condition (a precise definition will be given below) is either fulfilled for a circuit, or may be achieved by adding to the circuit small inductivities and capacities. A crucial role is here played by the concept of diode nonlinearity which will be explained below.

After providing the necessary theoretical backgroud, at the end of this paper we will give two typical examples. The first example illustrates how the LC-condition may be ensured "automatically" by our algorithm, while the second example is concerned with a circuit which does not satisfy an LC-condition but nevertheless leads to a diode nonlinearity.

1. Normal and tangent cones. In this section we recall some basic facts about normal and tangent cones. In what follows, we will always work in the Euclidean space $\mathbb{R}^m$ with scalar product $\langle x,y\rangle = x_1y_1 + \ldots + x_my_m$ and corresponding norm $\|x\|^2 = \langle x,x\rangle$.

A set $K \subseteq \mathbb{R}^m$ is a *cone* if $x,y \in K$ and $s,t \geqslant 0$ implies $sx + ty \in K$; so a cone is always convex, but not necessarily closed. Standard examples are the *positive octant* $\mathbb{R}^m_+ := \{x \in \mathbb{R}^m : x_1, \ldots, x_m \geqslant 0\}$ and the *negative octant* $\mathbb{R}^m_- := \{x \in \mathbb{R}^m : x_1, \ldots ,x_m \leqslant 0\}$; in particular, $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_- = (-\infty,0]$. The *adjoint cone* of a cone $K \subseteq \mathbb{R}^m$ is defined by

$$K^* := \{z \in \mathbb{R}^m : \langle y,z\rangle \leqslant 0 \text{ for all } y \in K\}. \tag{1}$$

The adjoint cone K^* is always closed, even if K is not. Moreover, it is easy to see that $K^{**} = K$ if K is closed.

Given a closed convex set $Q \subseteq \mathbb{R}^m$ and a point $x \in Q$, the *normal cone* to Q at x is defined by

$$N_Q(x) := \{y \in \mathbb{R}^m : \langle y,z - x\rangle \leqslant 0 \text{ for all } z \in Q\}, \tag{2}$$

while the *tangent cone* to Q at x is defined by

$$T_Q(x) := \{z \in \mathbb{R}^m : \langle y,z\rangle \leqslant 0 \text{ for all } y \in N_Q(x)\}. \tag{3}$$

A comparison with (1) shows that $T_Q(x) = N_Q(x)^*$, so a tangent cone is always closed. Moreover, we have $N_Q(x) = \{0\}$ and $T_Q(x) = \mathbb{R}^m$ if x is an interior point of Q; therefore the cones (2) and (3) are interesting only in case $x \in \partial Q$, the boundary of Q.

In some cases these cones may be calculated quite easily. For example, in the scalar case $Q = [a,b]$ we have

$$N_Q(a) = T_Q(b) = \mathbb{R}_-, \quad N_Q(b) = T_Q(a) = \mathbb{R}_+.$$

If $Q = \{x \in \mathbb{R}^m : \langle x,n\rangle \leqslant c\}$ is an affine halfspace, where $n \in \mathbb{R}^m$ is a normalized vector and $c \in \mathbb{R}$ is fixed, a straightforward calculation shows that $N_Q(x) = \{\lambda n : \lambda \geqslant 0\}$ and $T_Q(x) = \{x \in \mathbb{R}^m : \langle x,n\rangle \leqslant 0\}$ for any $x \in \partial Q = \{x \in \mathbb{R}^m : \langle x,n\rangle = c\}$.

The normal cone $N_x(Q)$ and the tangent cone $T_Q(x)$ have some "duality properties", the most important one being summarized in the following

Lemma 1. *For fixed x and Q, denote by $\nu_x : \mathbb{R}^m \to N_Q(x)$ the metric projection onto the normal cone* (2) *and by $\tau_x : \mathbb{R}^m \to T_Q(x)$ the metric projection onto the tangent cone* (3). *Then every $y \in \mathbb{R}^m$ admits an orthogonal decomposition of the form*

$$y = \nu_Q(y) + \tau_Q(y), \qquad \langle \nu_Q(y),\tau_Q(y)\rangle = 0. \tag{4}$$

Conversely, if $y = u + v$ for some $u \in N_Q(x)$ and $v \in T_Q(x)$, then $u = \nu_x(y)$ and $v = \tau_x(y)$.

The proof of Lemma 1 is standard and follows from the general fact that, given a convex set $Q \subset \mathbb{R}^m$ and an element $y \in \mathbb{R}^m$, the point of best approximation x of y in Q may be characterized by the variational inequality

$$\langle y - x, z - x\rangle \leqslant 0 \qquad (z \in Q).$$

In what follows, we will use the fact that in case of a cone K we have the equivalence

$$y \in N_K(x) \quad \Longleftrightarrow \quad x \in K,\ y \in K^*,\ \langle x,y \rangle = 0. \tag{5}$$

This follows from the maximal monotonicity of the multivalued map $x \mapsto N_K(x)$ in case of a cone. The basic definitions and results of the theory of cones may be found, e.g., in the textbooks [1] or [3].

2. Diode nonlinearities. Now we are going to define precisely what we mean by a DN-system and its solutions. Given an interval $I \subseteq \mathbb{R}$ and a closed convex set $Q \subseteq \mathbb{R}^m$, we will consider continuous maps $f : I \times Q \to \mathbb{R}^m$ in what follows.

Definition. A *system with diode nonlinearity* (or *DN-system*, for short) is a differential inclusion of the form

$$\dot{x} \in f(t,x) - N_Q(x) \tag{6}$$

involving the normal cone (2). A *solution* of (6) is an absolutely continuous function $x = x(t)$ satisfying

$$\dot{x}(t) \in f(t,x(t)) - N_Q(x(t)) \tag{7}$$

for almost all $t \in I$. □

Interestingly, DN-systems may be also formulated in the following different, but equivalent, form. Denoting as before by τ_x the metric projection onto the cone (3), which means that $\|y - \tau_x(y)\| = \mathrm{dist}(y,T_Q(x))$ for $y \in \mathbb{R}^m$, we may rewrite (6) in the form

$$\dot{x} = \tau_x f(t,x). \tag{8}$$

Let us briefly explain why (6) and (8) are equivalent. If $x = x(t)$ is absolutely continuous and satisfies

$$\dot{x}(t) = \tau_{x(t)} f(t,x(t)) \tag{9}$$

for almost all $t \in I$, from Lemma 1 we conclude that

$$\tau_{x(t)} f(t,x(t)) = f(t,x(t)) - \nu_{x(t)} f(t,x(t)) \in f(t,x(t)) - N_Q(x(t)),$$

where ν_x is the metric projection onto the cone (2). Conversely, suppose that $x = x(t)$ is absolutely continuous and satisfies (7) almost everywhere. Choose a point $u \in N_Q(x(t))$ such that $\dot{x}(t) = f(t,x(t)) - u$. Then $\dot{x}(t)$ and u are orthogonal, by Lemma 1, since $\dot{x}(t) \in T_Q(x(t))$. Now, the representation $f(t,x(t)) = u + \dot{x}(t)$ and the orthogonality of u and $\dot{x}(t)$ shows that $u = \nu_{x(t)} f(t,x(t))$ and $\dot{x}(t) = \tau_{x(t)} f(t,x(t))$, again by Lemma 1, and the last equality is precisely (9).

To treat DN-systems in the representation (6) we have to analyze the properties of the multivalued map $N_Q : x \mapsto N_Q(x)$ (which is sometimes called the *DN-operator generated by* Q in the literature). For instance, one may show that N_Q has a *closed graph* in the sense that, if $x_n \in Q$ satisfies $x_n \to x$ as $n \to \infty$, and $y_n \in N_Q(x_n)$ satisfies $y_n \to y$ as $n \to \infty$, then $y \in N_Q(x)$. We point out, however, that the multivalued map $T_Q : x \mapsto T_Q(x)$ in general does not have a closed graph.

The advantage of the representation (8) is of course that it does not involve multivalued maps. On the other hand, (8) is *not* an ordinary differential equation, since it contains the projection operator τ_x, which makes its treatment surprisingly difficult.

Before starting the application-oriented part of this paper, we need another lemma which shows how didode nonlinearities behave under linear transformations.

Lemma 2. *Let* $K \subseteq \mathbb{R}^m$ *be a cone,* $x \in K$*,* $u \in \mathbb{R}^n$*, and* $A : \mathbb{R}^m \to \mathbb{R}^n$ *a linear operator. Then the equivalence*

$$u \in N_{A(K)}(Ax) \quad \Longleftrightarrow \quad A^* u \in N_K(x) \tag{10}$$

holds, where A^* denotes the adjoint of A.

Proof. The relation on the left hand side of (10) means, by (5), that

$$Ax \in A(K), \quad u \in A(K)^*, \quad \langle Ax,u\rangle = 0.$$

But this is obviously equivalent to

$$x \in K, \quad A^*u \in K^*, \quad \langle x,A^*u\rangle = 0$$

which is precisely the right hand side of (10). □

3. Electrical circuits. Consider a *diode converter* which represents an electrical circuit consisting of m ideal diods. We will assume that all knots of this circuit are *inputs*, i.e., contacts through which the diode converter may be joined with other circuits. We enumerate the knots (inputs) in a predetermined way by $0,1,\ldots,n$. In the j-th diode, we denote the corresponding current by x_j and the corresponding voltage (from the anode to the cathode) by y_j. Furthermore, the *incoming current*, i.e., the current which flows from an outer circuit to the diode converter, through the k-th input will be denoted by i_k, the *incoming voltage* i.e., the voltage between the k-th input and the 0-th input, by u_k $(k = 1,2,\ldots,n)$. With this notation, the so-called *Volt-Ampère characteristic* of the ideal diode may be written in the form $y \in N_{\mathbb{R}^m_+}(x)$, which means by (5) that

$$x \in \mathbb{R}^m_+, \ y \in \mathbb{R}^m_-, \ \langle x,y\rangle = 0. \tag{11}$$

In Figure 1 we have sketched an example of two circuits, where Figure 1 (a) contains only the diodes, while in Figure 1 (b) the branches of the constructed diode converter are sketched by dotted lines with arrows.

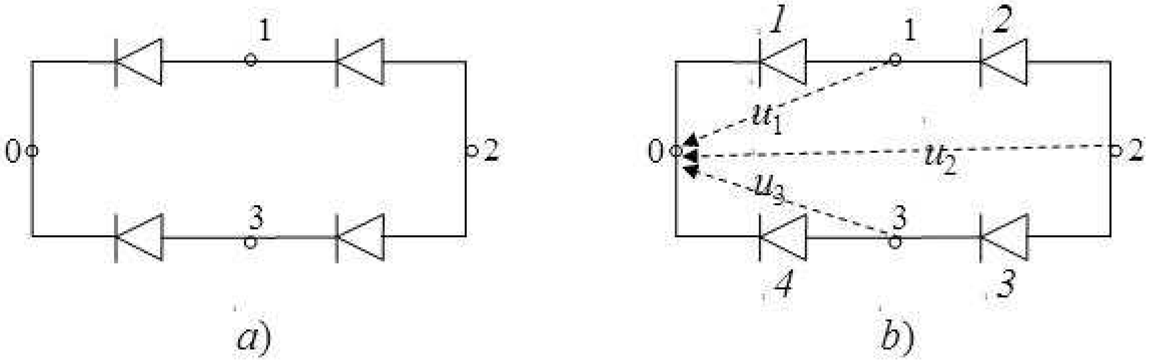

Figure 1.

The relation between the anode current vector $x = (x_1,x_2,\ldots,x_m)$ and the input current vector $i = (i_1,i_2,\ldots,i_n)$ of a general electrical circuit is given by $Ax = i$, where the $(n \times m)$-matrix $A = (a_{kj})_{k,j}$ has the entries

$$a_{kj} = \begin{cases} 1 & \text{if the } j\text{-th diode anode is joined with the } k\text{-th knot,} \\ -1 & \text{if the } j\text{-th diode cathode is joined with the } k\text{-th knot,} \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

We do not take into account the rule for the 0-th knot, since it follows from the corresponding rules for the other knots and our assumption that the sum of all input currents is zero. In the following Proposition, we denote by

$$A^j := (a_{1j},a_{2j},\ldots,a_{nj})^T \qquad (j = 1,2,\ldots,m) \tag{13}$$

the j-th column of the matrix $A = (a_{kj})_{k,j}$ with elements (12).

Proposition. *Let $K = \mathbb{R}^m_+$ be the cone of all m-tuples with nonnegative coordinates. Then the input voltage vector u and input current vector i are related by the equality*

$$u \in N_{A(K)}(i), \tag{14}$$

where $N_{A(K)}$ denotes the DN-operator generated by $A(K)$.

Proof. Note first that the cone $A(K)$ contains all elements generated by the columns $A^1,\ldots,A^m$ of the matrix A, i.e.

$$A(K) = \left\{s_1A^1 + \ldots + s_mA^m : s_1,\ldots,s_m \geqslant 0\right\},$$

see (13). Fix x and y satisfying (11), which means that $y \in N_K(x)$, by (5). Let $k(j,+)$ be the number of the knot which is joined to the anode of the j-th diode, and let $k(j,-)$ be the analogous number for the cathode. Then

$$y_j = u_{k(j,+)} - u_{k(j,-)} \qquad (j = 1,2,\ldots,m).$$

In the j-th column in (13), only the entries with index $k(j,+)$ or $k(j,-)$ can be different from zero. If one of these indices is zero, then the corresponding input voltage is also zero. From this we conclude that $y = A^*u$, where A^* is the adjoint matrix to A. So Lemma 2 implies that $u \in N_{A(K)}(Ax) = N_{A(K)}(i)$ as claimed. □

4. Main result. Consider a connected electrical circuit which contains the usual elements: sources S, resistances R, capacities C, inductances L, and diodes D.

In order to illustrate the Kirchhoff rules in the theory of electrical circuits, it is common to draw a certain *graph tree* which contains all knots, but no contour. The branches (elements) which are not contained in the tree all called *connectivity branches*; each of them closes precisely one *pricipal contour* which includes, apart from the given branch, only branches of the tree. Conversely, every branch of the tree forms precisely one *principal cross-section*, i.e., a choice of branches which contain, apart from the given branch of the tree, all those connectivity branches whose principal contours include the given branch.

We denote by $\bar{U}$ the voltage vector and by $\bar{I}$ the current vector in the branches of a circuit, while by U we denote the voltage vector and by I the current vector in the branches of the tree. Then the *principal contour equation* $\bar{U} = MU$ and the *principal cross-section equation* $I = -M^*\bar{I}$ are related by the same matrix M and its adjoint M^* (see [2] or [4]).

We denote by D_1 the set of all diodes which are connected to the circuit by capacities (i.e., which form a contour together with one of the involved capacities), and by k_1 the number of elements of D_1. The set of all the other diodes is denoted by D_2, the number of its elements by k_2. Enumerating the diodes of D_1 and D_2 separately in an arbitrary order, we construct a *diodic converter* $\mathbf{D}$ in the following way. All elements of D_1 are considered as branches of $\mathbf{D}$ and denoted by $\mathbf{D_1}$, while all elements of D_2 are considered as branches of $\mathbf{D}$, and denoted by $\mathbf{D_2}$.

Consider the column vector

$$x := (y_1,y_2,\ldots,y_{k_1},x_1,x_2,\ldots,x_{k_2})^T \tag{15}$$

consisting of the k_1 anode voltages of the diode D_1 and the k_2 anode currents of the diode D_2, together with the row vector

$$y := (x_1,x_2,\ldots,x_{k_1},y_1,y_2,\ldots,y_{k_2}) \tag{16}$$

consisting of the k_1 anode currents of the diode D_1 and the k_2 anode voltages of the diode D_2. The vector (15) belongs to the cone $\mathbb{R}^{k_1}_- \times \mathbb{R}^{k_2}_+$, the vector (16) belongs to the cone $\mathbb{R}^{k_1}_+ \times \mathbb{R}^{k_2}_-$, and each of these cones is adjoint to the other. In particular, the elements x and y are mutually orthogonal, so they are related by means of a didodic nonlinearity operator.

Denote by

$$v := (u_1,u_2,\dots,u_{k_1},i_1,i_2,\dots,i_n)^T$$

the vector consisting of the anode voltages at the diodic converter $\mathbf{D_1}$ and the currents at the inputs of the diodic converter $\mathbf{D_2}$, and by

$$u := (i_1,i_2,\dots,i_{k_1},u_1,u_2,\dots,u_n)$$

the vector consisting of the currents at the diodic converter $\mathbf{D_1}$ and the voltages at the branches of the diodic converter $\mathbf{D_2}$. Then the dependence of v on x may be expressed through the equation

$$v = \begin{pmatrix} E & O \\ O & A \end{pmatrix} x,$$

with A denoting the $(k_1 \times k_1)$-matrix with columns (13), E the unit matrix of rank k_1, and O the zero matrix of order k_2. Likewise, the dependence of y on u may be expressed through the equation

$$y = \begin{pmatrix} E & O \\ O & A^* \end{pmatrix} u.$$

Consequently, our Proposition implies that u and v are connected through the corresponding DN-operator. In what follows, we will assume the following crucial

- **LC-condition:** *Every path of the elements S, R, C, and L which joins two inputs from the part $\mathbf{D_2}$ of the didodic converter contains at least one inductivity.*

Now we split all elements of the circuit into the 6 groups C, $\mathbf{D_2}$, S, R, L and $\mathbf{D_1}$. In the group C we first enumerate the capacities which are parallel-joined to the diodes of $\mathbf{D_1}$, then all the other capacities in arbitrary order. In the groups S, R and L we enumerate all elements in arbitrary order. Finally, in the groups $\mathbf{D_2}$ and $\mathbf{D_1}$ we keep the original order imposed in the construction of $\mathbf{D} = \mathbf{D_2} \cup \mathbf{D_2}$. Now we construct the circuit tree by resetting the groups and their elements according to the chosen enumeration, where we connect every time an element to the tree when it does not form a contour for the previously connected elements.

From this description and the LC-conditions stated above it follows that all elements of $\mathbf{D_2}$ belong to the tree, while all elements of $\mathbf{D_1}$ are branches which close a contour with one of the capacities.

To illustrate our construction, we have sketched in Figure 2 two circuits which both satisfy the LC-condition. According to our algorithm, we have first joined in Figure 2 (a) the branches 1, 2 and 3 of the diode converter on the 4 diodes which are sketched in light grey. Afterwards we have joined the branch of the voltage source E, and then the branches of the resistances R_1 and R_2. The branches L_1 and L_2 here belong to the circuit, while the sets C and $\mathbf{D_1}$ described above are empty.

The circuit sketched in Figure 2 (b) consists only of one capacity branch C, while all the other branches are connected. For this circuit the sets L and $\mathbf{D_2}$ are empty.

Once we have constructed the tree in this way, we enumerate separately in every group the tree elements and the connecting elements (not in the tree) in the old order. Moreover, we overline the voltages and currents of all connecting branches, observing that this does not change the emumeration in the diodic converter.

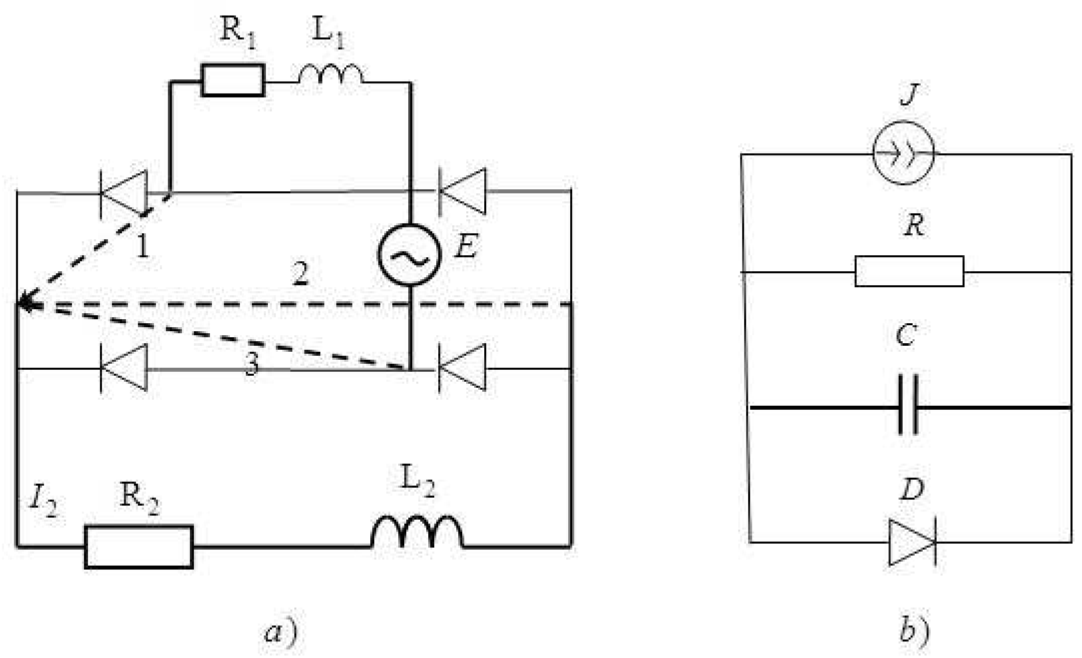

Figure 2.

Then the equations of the principal contours and principal cross-sections of the resulting tree have the form

$$\begin{cases} \bar{u}_C = M_{11}u_C, \\ \bar{u}_S = M_{31}u_C + M_{33}u_S, \\ \bar{u}_R = M_{41}u_C + M_{43}u_S + M_{44}u_R, \\ \bar{u}_L = M_{51}u_C + M_{52}u_{\mathbf{D_1}} + M_{53}u_S + M_{54}u_R + M_{55}u_L, \\ \bar{u}_{\mathbf{D_2}} = M_{61}u_C, \end{cases} \tag{17}$$

and

$$\begin{cases} i_C = -M_{11}^*\bar{i}_C - M_{31}^*\bar{i}_S - M_{41}^*\bar{i}_R - M_{51}^*\bar{i}_L - M_{61}^*\bar{i}_{\mathbf{D_2}}, \\ i_S = -M_{33}^*\bar{i}_S - M_{43}^*\bar{i}_R - M_{53}^*\bar{i}_L, \\ i_R = -M_{44}^*\bar{i}_R - M_{54}^*\bar{i}_L, \\ i_L = -M_{55}^*\bar{i}_L, \\ i_{\mathbf{D_1}} = -M_{52}^*\bar{i}_L, \end{cases} \tag{18}$$

respectively. Here M_{ij} and M_{ij}^* are stationary (i.e., time-independent) matrices which contain, as the matrix A with entries (12), only the elements 0, 1 and -1, according to whether or not the corresponding element belongs to the closed contour, and in which direction. In matrix form the linear systems (17) and (18) read

$$\begin{pmatrix} \bar{u}_C \\ \bar{u}_S \\ \bar{u}_R \\ \bar{u}_L \\ \bar{u}_{\mathbf{D_2}} \end{pmatrix} = \begin{pmatrix} M_{11} & 0 & 0 & 0 & 0 \\ M_{31} & M_{33} & 0 & 0 & 0 \\ M_{41} & M_{43} & M_{44} & 0 & 0 \\ M_{51} & M_{53} & M_{54} & M_{55} & M_{52} \\ M_{61} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_C \\ u_S \\ u_R \\ u_L \\ u_{\mathbf{D_1}} \end{pmatrix}$$

and

$$\begin{pmatrix} i_C \\ i_S \\ i_R \\ i_L \\ i_{\mathbf{D_1}} \end{pmatrix} = - \begin{pmatrix} M_{11}^* & M_{31}^* & M_{41}^* & M_{51}^* & M_{61}^* \\ 0 & M_{33}^* & M_{43}^* & M_{53}^* & 0 \\ 0 & 0 & M_{44}^* & M_{54}^* & 0 \\ 0 & 0 & 0 & M_{55}^* & 0 \\ 0 & 0 & 0 & M_{52}^* & 0 \end{pmatrix} \begin{pmatrix} \bar{i}_C \\ \bar{i}_S \\ \bar{i}_R \\ \bar{i}_L \\ \bar{i}_{\mathbf{D_2}} \end{pmatrix},$$

respectively. For the sake of completeness of the mathematical description, we also add the equations for the inductivity, capacity, and resistance of the circuit

$$\begin{cases} \bar{L}\,\bar{i}_L' = \bar{u}_L, \ \bar{C}\,\bar{u}_C' = \bar{i}_C, \ \bar{u}_R = \bar{R}\,\bar{i}_R, \\ Li_L' = u_L, \ Cu_C' = i_C, \ u_R = Ri_R, \end{cases} \tag{19}$$

where $\bar{L}$, L, $\bar{C}$, C, $\bar{R}$, and R are diagonal matrices with positive entries.

We assume that u_S and $\bar{i}_S$ are known. Physically, this means that all voltage sources belong to the tree, while all current sources are connecting branches; in the opposite case the scheme may be contradictory.

Let us now study and simplify the transformation of the systems (17) and (18). First we remove $\bar{i}_C$ and u_L. Taking derivatives in the first equation of (17) and multiplying by $\bar{C}$, we obtain

$$\bar{i}_C = \bar{C}\,\bar{u}_C' = \bar{C}M_{11}C^{-1}Cu_C' = \bar{C}M_{11}C^{-1}i_C,$$

where we have used the second and fifth equality in (19). Putting this expression for $\bar{i}_C$ into the first equation in (18) yields

$$i_C + M_{11}^*\bar{C}M_{11}C^{-1}i_C = -M_{31}^*\bar{i}_S - M_{41}^*\bar{i}_R - M_{51}^*\bar{i}_L - M_{61}^*\bar{i}_{\mathbf{D_2}}. \tag{20}$$

Similarly, taking derivatives in the fourth equation of (18) and multiplying by L, we obtain

$$u_L = Li_L' = -LM_{55}^*\bar{L}^{-1}\bar{L}\bar{i}_L' = -LM_{55}^*\bar{L}^{-1}\bar{u}_L,$$

we have used the first and fourth equality in (19). Putting this expression for u_L into the fourth equation in (17) yields

$$\bar{u}_L + M_{55}LM_{55}^*\bar{L}^{-1}\bar{u}_L = M_{51}u_C + M_{52}u_{\mathbf{D_1}} + M_{53}u_S + M_{54}u_R. \tag{21}$$

Using the shortcut $A := C + M_{11}^*\bar{C}M_{11}$ and $B := \bar{L} + M_{55}LM_{55}^*$, and applying the operators AC^{-1} and $B\bar{L}^{-1}$ to the inductivity equation (20) and capacity equation (21), respectively, we end up with

$$Au_C' = AC^{-1}i_C = -M_{31}^*\bar{i}_S - M_{41}^*\bar{i}_R - M_{51}^*\bar{i}_L - M_{61}^*\bar{i}_{\mathbf{D_2}} \tag{22}$$

and

$$B\bar{i}_L' = B\bar{L}^{-1}\bar{u}_L = M_{51}u_C + M_{52}u_{\mathbf{D_1}} + M_{53}u_S + M_{54}u_R, \tag{23}$$

respectively. Now we remove $\bar{i}_R$ and u_R from the systems (17) and (18). To determine $\bar{i}_R$ we use the third equation in (17) and third equation in (18) and obtain

$$(\bar{R} + M_{44}RM_{44}^*)\bar{i}_R = M_{41}u_C + M_{43}u_S - M_{44}RM_{54}^*\bar{i}_L.$$

Being symmetric and positive definite, the matrix $S := \bar{R} + M_{44}RM_{44}^*$ is invertible, and so

$$\bar{i}_R = S^{-1}M_{41}u_C + S^{-1}M_{43}u_S - S^{-1}M_{44}RM_{54}^*\bar{i}_L. \tag{24}$$

To determine the voltage and the other currents we put (24) into the equation for the resistance and get

$$\bar{u}_R = \bar{R}S^{-1}M_{41}u_C + \bar{R}S^{-1}M_{43}u_S - \bar{R}S^{-1}M_{44}RM_{54}^*\bar{i}_L.$$

Likewise, putting (24) into the third equation from (18) yields

$$i_R = -M_{44}^*S^{-1}M_{41}u_C - M_{44}^*S^{-1}M_{43}u_S + (M_{44}^*S^{-1}M_{44}R - E)M_{54}^*\bar{i}_L,$$

where E denotes as before the unit matrix. Finally, applying the matrix R we arrive at

$$u_R = -RM_{44}^*S^{-1}M_{41}u_C - RM_{44}^*S^{-1}M_{43}u_S + R(M_{44}^*S^{-1}M_{44}R - E)M_{54}^*\bar{i}_L,$$

where we have used the last equality in (19). Introducing the vectors

$$x := \begin{pmatrix} \bar{i}_L \\ u_C \end{pmatrix}, \quad u := \begin{pmatrix} \bar{i}_{\mathbf{D_2}} \\ u_{\mathbf{D_1}} \end{pmatrix}, \quad v := \begin{pmatrix} \bar{u}_{\mathbf{D_2}} \\ i_{\mathbf{D_1}} \end{pmatrix}, \quad y := \begin{pmatrix} \bar{i}_S \\ u_S \end{pmatrix} \tag{25}$$

and the matrices

$$A_1 := \begin{pmatrix} B & O \\ O & A \end{pmatrix}, \; A_2 := \begin{pmatrix} M_{54}R(M_{44}^*S^{-1}M_{44}R - E)M_{54}^* & M_{51} - M_{54}RM_{44}^*S^{-1}M_{41} \\ M_{41}^*S^{-1}M_{44}RM_{54}^* - M_{51}^* & -M_{41}^*S^{-1}M_{41} \end{pmatrix},$$

$$A_3 := \begin{pmatrix} O & -M_{52} \\ M_{61}^* & O \end{pmatrix}, \; A_4 := \begin{pmatrix} O & M_{53} - M_{54}RM_{44}^*S^{-1}M_{43} \\ -M_{31}^* & -M_{41}^*S^{-1}M_{43} \end{pmatrix},$$

we may write (22) and (23) more concisely in the form

$$A_1x' = A_2x - A_3u + A_4y, \tag{26}$$

and the last equations from (17) and (18) more concisely in the form

$$v = A_3^*x.$$

Let us recall that the function $y = y(t)$ is known; this function defines the action of the voltage and current sources. Moreover, we point out that the matrix A_1 is symmetric and positive definite, so the matrices $A_1^{1/2}$ and $A_1^{-1/2}$ are well-defined. Putting

$$X := A_1^{1/2}x, \quad U := A_1^{-1/2}A_3u, \quad f(t,X) := A_1^{-1/2}A_2A_1^{-1/2}X + A_1^{-1/2}A_4y \tag{27}$$

we see that $v = A_3^*A_1^{-1/2}X$. Applying $A_1^{-1/2}$ to both sides of (26) we obtain

$$\begin{aligned} X' = A_1^{1/2}x' = A_1^{-1/2}A_1x' = A_1^{-1/2}A_2x - A_1^{-1/2}A_3u + A_1^{-1/2}A_4y \\ = A_1^{-1/2}A_2A_1^{-1/2}X - U + A_1^{-1/2}A_4y = f(t,X) - U. \end{aligned} \tag{28}$$

The diode converter vectors u and v are connected, as we already observed, by the corresponding DN-operator. So from the above Proposition we may conclude that the vector $U = A_1^{-1/2}A_3u$ in (28) is related to X through the diode nonlinearity N_K generated by some cone K. Consequently, (28) may be rewritten as DN-system

$$X' \in f(t,X) - N_K(X), \tag{29}$$

as we have shown before. We summarize our discussion with the following

Theorem. *Suppose that the physical model for the diode converter of an electrical circuit may be represented in such a way that it satisfies the LC-condition. Then the mathematical model for the circuit may be represented as DN-system.*

This theorem is not only of theoretical interest. In fact, as soon as we are able to solve the differential inclusion (29), we may calculate all voltages and currents in the corresponding circuit.

5. Examples. In this final section we first illustrate the previous construction by means of an example, where the LC-condition which was crucial for deriving the differential inclusion (29) is satisfied. Afterwards we give an example which shows that this condition is sufficient, but not necessary.

Example 1. Consider again the circuit sketched in Figure 2 (a). Since this circuit contains two inductances, two resistances, and one diode, the corresponding vectors occurring in this circuit are

$$\bar{u}_L = \begin{pmatrix} \bar{u}_{L_1} \\ \bar{u}_{L_2} \end{pmatrix}, \ \bar{i}_L = \begin{pmatrix} \bar{i}_{L_1} \\ \bar{i}_{L_2} \end{pmatrix}, \ u_R = \begin{pmatrix} u_{R_1} \\ u_{R_2} \end{pmatrix}, \ i_R = \begin{pmatrix} i_{R_1} \\ i_{R_2} \end{pmatrix}.$$

Moreover,

$$u_{D_2} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \ i_{D_2} = \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix}, \ u_S = E(t).$$

The only principal contour equation occurring here in (17) is

$$\bar{u}_L = M_{52}u_{D_2} + M_{53}E(t) + M_{54}u_R,$$

where

$$M_{52} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad M_{53} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad M_{54} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{30}$$

while the three principal cross-section equations in (18) occurring here have the form

$$i_S = -M_{53}^*\bar{i}_L, \quad i_R = -M_{54}^*\bar{i}_L, \quad i_{D_2} = -M_{52}^*\bar{i}_L.$$

All other matrices in (17) or (18) are zero. Let us now see how the differential equations in (19) look like in this case. Since

$$L_1\bar{i}'_{L_1} = \bar{u}_{L_1}, \ L_2\bar{i}'_{L_2} = \bar{u}_{L_2}, \ u_{R_1} = R_1 i_{R_1}, \ u_{R_2} = R_2 i_{R_2},$$

the first and last equations in (19) hold with

$$\bar{L} = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \quad R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}.$$

The matrix $A = C + M_{11}^*\bar{C}M_{11}$ does not occur, but $B = \bar{L} + M_{55}LM_{55}^* = \bar{L}$, since $M_{55} = O$. Consequently,

$$A_1 = \bar{L}, \ A_2 = -R, \ A_3 = -M_{52}, \ A_4 = M_{53},$$

by (29). Taking into account (26) and the fact that $x = \bar{i}_L$ and $y = u_S = E(t)$, the transformation (27) reads

$$X = A_1^{1/2}x = \begin{pmatrix} L_1^{1/2}\bar{i}_{L_1} \\ L_2^{1/2}\bar{i}_{L_2} \end{pmatrix}, \quad U = A_1^{-1/2}A_3u = \begin{pmatrix} L_1^{-1/2}(u_1 - u_3) \\ L_2^{-1/2}u_2 \end{pmatrix},$$

and

$$f(t,X) = A_1^{-1/2} A_2 A_1^{-1/2} X + A_1^{-1/2} A_4 y$$

$$= \begin{pmatrix} L_1^{-1/2} & 0 \\ 0 & L_2^{-1/2} \end{pmatrix} \begin{pmatrix} -R_1 & 0 \\ 0 & -R_2 \end{pmatrix} \begin{pmatrix} L_1^{-1/2} & 0 \\ 0 & L_2^{-1/2} \end{pmatrix} \begin{pmatrix} L_1^{1/2} \bar{i}_{L_1} \\ L_2^{1/2} \bar{i}_{L_2} \end{pmatrix}$$

$$+ \begin{pmatrix} L_1^{-1/2} & 0 \\ 0 & L_2^{-1/2} \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} E(t) = \begin{pmatrix} -R_1 L_1^{-1} & 0 \\ 0 & -R_2 L_2^{-1} \end{pmatrix} X + \begin{pmatrix} -E(t) L_1^{-1/2} \\ 0 \end{pmatrix}.$$

So from our main result we conclude that the circuit in Figure 2 (a) may be described by the DN-system (29) for given alimentation inputs $E(t)$. The cone K in (29) may be described explicitly. Let K_D be the cone consisting of all elements of the form

$$z := \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \begin{pmatrix} \xi_1 - \xi_2 \\ \xi_2 - \xi_3 \\ \xi_4 - \xi_3 \end{pmatrix},$$

where the vector $(\xi_1,\xi_2,\xi_3,\xi_4)$ runs over the positive octant $\mathbb{R}^4$. Then K is the adjoint cone to $-A_1^{-1/2} M_{52} K_D^*$, where M_{52} is the first matrix in (30). □

Example 2. The circuit sketched in Figure 3 below does not satisfy the LC-condition, because the three paths which contain the elements J_1 and J_2 and join the didodic converter from $\mathbf{D_2}$ do not include an inductivity.

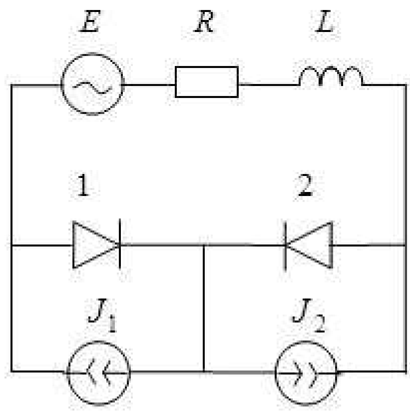

Figure 3.

The equations for the voltage drop in the contour $\{E,R,L,D_2,D_1\}$ has here the form

$$u_R + u_L + u_{D_2} - u_{D_1} = u_E, \tag{31}$$

where $u_E = u_E(t)$ is some given time-dependent function. We write

$$i := i_R = i_L = i_{D_2} - J_2 = -i_{D_1} + J_1,$$

with given constants J_1 and J_2. Expressing the voltages u_R and u_L in (31) through the relations $u_R(t) = i(t)R(t)$ and $u_L(t) = i'(t)L(t)$, we arrive at

$$iR + i'L + u_{D_2} - u_{D_1} = u_E. \tag{32}$$

Since the anode currents $i_{D_1} = J_1 - i$ and $i_{D_2} = J_2 + i$ of the diodes D_1 and D_2, respectively, assume only nonnegative values, we get $-J_2 \leqslant i \leqslant J_1$; so for the correct performance of the circuit we require that $J_1 + J_2 \geqslant 0$ in order guarantee that $Q := [-J_2, J_1] \neq \varnothing$.

Now we distinguish three cases for the position of i in Q. If $-J_2 < i < J_1$ then $i_{D_1} > 0$ and $i_{D_2} > 0$, so $u_{D_1} = u_{D_2} = 0$. If $i = -J_2$ then $i_{D_1} > 0$, $i_{D_2} = 0$, $u_{D_1} = 0$, and $u_{D_2} \leqslant 0$. Finally, if $i = J_1$ then $i_{D_1} = 0$, $i_{D_2} > 0$, $u_{D_1} \leqslant 0$, and $u_{D_2} = 0$. In any case the vector $u := u_{D_2} - u_{D_1}$ belongs to $N_Q(i)$.

Writing (32) in the form

$$i'(t) = \frac{u_E(t)}{L(t)} - i(t)\frac{R(t)}{L(t)} - \frac{u(t)}{L(t)} =: f(t,i(t)) - \frac{u(t)}{L(t)},$$

and observing that also u/L belongs to $N_Q(i)$, since L is positive, we end up with the differential inclusion

$$i' \in f(t,i) - N_Q(i)$$

which is precisely of the form (6) (or (29)). In other words, the mathematical model of the circuit sketched in Figure 3 may be represented as DN-system. □

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