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REVIEW OF ATTRACTORS FOR A MODEL OF MOTION OF WEAK AQUEOUS POLYMER SOLUTIONS

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Abstract: the aim of this paper is to demonstrate how the approximating topological method can be effectively combined with the theory of attractors of trajectory spaces in problems of fluid mechanics. We consider the model of motion of weak aqueous polymer solutions and prove that it has the minimal trajectory attractor and the global one. Then we prove that the attractors of approximating problem converge to the attractors of the unperturbed one.

Key words and phrases: Non-Newtonian fluids, approximating topological method, trajectory space, trajectory attractor, global attractor, convergence of attractors.

ОБЗОР АТТРАКТОРОВ ДЛЯ МОДЕЛИ ДВИЖЕНИЯ СЛАБЫХ ВОДНЫХ РАСТВОРОВ ПОЛИМЕРОВ

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Аннотация: целью данной работы является демонстрация совместного использования аппроксимационно-топологического метода и теории аттракторов траекторных пространств в задачах неньютоновской гидродинамики. В статье рассматривается одна математическая модель неньютоновской гидродинамики - модель движения слабо концентрированных водных растворов полимеров. Для исследования рассматриваемой модели рассматривается аппроксимационная задача, разрешимость которой доказывается на основе теории степени Лере-Шаудера и априорных оценок решений. На этой основе доказывается существование минимального траекторного и глобального аттракторов исходной задачи. Также доказывается, что аттракторы аппроксимационной задачи сходятся к аттракторам исходной модели.

Ключевые слова: неньютонова жидкость, аппроксимационно-топологический метод, траекторные пространства, траекторный аттрактор, глобальный аттрактор, сходимость аттракторов.

1. EQUATIONS OF MOTION

We illustrate the application of the approximating topological approach to problems of fluid mechanics with the autonomous initial boundary problem for the mathematical model of motion of weak aqueous polymer solutions. We prove that this problem has trajectory and global attractors.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary ($n = 2, 3$). Consider the initial boundary problem

$$\frac{\partial v}{\partial t} - \nu \Delta v + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \varkappa \frac{\partial \Delta v}{\partial t}$$

$$-2\varkappa \operatorname{Div} \left(\sum_{i=1}^{n} v_i \frac{\partial \mathcal{E}(v)}{\partial x_i} \right) + \operatorname{grad} p = f, \quad (x,t) \in \Omega \times (0, +\infty), \tag{1.1}$$

$$\operatorname{div} v = 0, \quad (x,t) \in \Omega \times (0, +\infty), \tag{1.2}$$

$$v\Big|_{\partial\Omega} = 0, \quad t \in (0, +\infty), \tag{1.3}$$

$$v\Big|_{t=0} = a, \quad x \in \Omega. \tag{1.4}$$

Here $v(x,t)$ is the vector of velocity of the particle that is situated at the point x at the moment of time t; $p(x,t)$ is the preassure of the fluid at the point x at the moment of time t; $f(x,t)$ is the vector of body force; $\mathcal{E} = (\mathcal{E}_{ij})$ is the strain velocity tensor, i. e. a symmetric matrix of order n with the components

$$\mathcal{E}_{ij} = \mathcal{E}_{ij}(v) = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right);$$

$\nu > 0$ is the kinematic coefficient of viscosity , $\varkappa > 0$ is the retardation time, a is a vector field on Ω, which belongs to a functional space to be specified below. Unknown functions are v and p.

Equations (1.1) and (1.2) constitute the *mathematical model of motion of weak aqueous polymer solutions.* Equation (1.1) coresponds to the constitutive law

$$\sigma = 2\nu \left(\mathcal{E} + \varkappa \nu^{-1} \frac{d\mathcal{E}}{dt} \right),$$

which establishes a relation between the deviator of the rate-of-strain tensor σ and the strain velocity tensor $\mathcal{E}$. This constitutive law was suggested in [1] on the basis of research [2].

Equation (1.3) is the boundary non-slip condition. We regard the coefficients involved in (1.1)–(1.3) and the external force f as fixed. On the contrary, the function a defining the initial condition (1.4) can be arbitrarily chosen in a functional space. Thus we obtain a set of weak solutions and use them so as to construct the trajectory space.

The solvability of problem (1.1)–(1.4) is treated in [3] and [4]. In the case of this problem neither the global solvability in the strong sense nor the uniqueness of the weak solution have been proved. Consequently, it is impossible to use the classical semigroup approach to attractors.

2. FUNCTIONAL SPACES AND NOTATIONS

We use standard notations for the spaces of integrable functions and the Sobolev spaces.

Now we describe the scale of spaces V^α (see [6], [5]).

Let $\mathcal{V}$ denote the set of smooth nondivergent fector fields whose supports lie in Ω.

Let V^0 and V^1 be the closures of $\mathcal{V}$ in $(L_2(\Omega))^n$ and $(H^1(\Omega))^n = (W_2^1(\Omega))^n$ correspondingly. Then V^0 endowed with the L_2 product $(\cdot,\cdot)$ is a Hilbert space. Let $\|\cdot\|_0$ denote the Hilbert norm. The space V^1 is Banach with respect to the norm

$$\|u\|_1 = \|\nabla u\|_{L_2} := \left(\sum_{i,j=1}^{n} \left\| \frac{\partial u_i}{\partial x_j} \right\|^2_{L_2(\Omega)} \right)^{\frac{1}{2}}, \tag{2.1}$$

Here ∇u denotes the Jacobi matrix of the vector function u. The norm (2.1) is equivalent to the norm induced by $(H^1(\Omega))^n$. This follows from the Friedrich's inequality

$$\|u\|_{L_2(\Omega)} \leqslant K_0 \|\nabla u\|_{(L_2(\Omega))^n} \qquad (u \in H_0^1(\Omega)), \tag{2.2}$$

where the constant K_0 does not depend on u. Note that for $n = 2, 3$ the embedding $V^1 \subset (L_4(\Omega))^n$ is compact. This follows from Sobolev's embedding theorems.

Put $V^2 = V^1 \cap (H^2(\Omega))^n$.

Consider the well-known Weyl decomposition (see [7]) of $(L_2(\Omega))^n$ into the orthogonal sum

$$(L_2(\Omega))^n = V^0 \oplus \nabla H^1(\Omega).$$

Let $\pi\colon (L_2(\Omega))^n \to V^0$ be the orthoprojector. Consider the operator

$$A = -\pi\Delta \tag{2.3}$$

defined on V^2. It is known that A can be extended to a positive self-addjoint operator in V^0 with the compact inverse operator. Hence A has countably many eigenvalues

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots;$$

Let e_k denote associated eigenfunctions. The vector functions e_k $(k = 1, 2, \ldots)$ are smooth.

Consider the set

$$E_\infty = \left\{ v = \sum_{k=1}^{m} v_k e_k \colon m \in \mathbb{N},\ v_k \in \mathbb{R} \right\}$$

(here m depends on v) and for any $\alpha \in \mathbb{R}$ define the space V^α as the completion of E_∞ with respect to the norm

$$\|v\|_\alpha = \Big(\sum_{k=1}^{\infty} \lambda_k^\alpha |v_k|^2 \Big)^{1/2}. \tag{2.4}$$

This norm is generated by the scalar product $(\cdot,\cdot)_\alpha$. The space V^α is Hilbert with respect to this scalar product.

It can be shown that for $\alpha = 0, 1, 2$ the construction described above leads to the same spaces V^0, V^1, and V^2 and norms $\|\cdot\|_0$ and $\|\cdot\|_1$ as introduced at the beginning.

If $\alpha \geqslant 0$, the space V^α consists of square-integrable functions belonging to V^0. If $\alpha < 0$, the space V^α is wider than V^0, i. e. it contains ideal elements. Let $\beta \geqslant 0$ and let $(V^\beta)^*$ be the conjugate space of V^β. Then the space $(V^\beta)^*$ is isometric to $V^{-\beta}$. We identify these spaces.

In case $\alpha \geqslant 0$ we have continuous embedding $V^\alpha \subset (H^\alpha(\Omega))^n$, and the norm $\|\cdot\|_\alpha$ is equivalent to the norm induced in V^α by $(H^\alpha(\Omega))^n$ (see [6]). For $\alpha > \beta \geqslant 0$ the embedding $V^\alpha \subset V^\beta$ is compact.

We shall be mostly concerned with the spaces V^0, V^1, V^3 and their conjugates. It can be proved [6] that for $\alpha = 1$ the norm (2.4) is given by (2.1), and for $\alpha = 3$ we have

$$\|v\|_3 = \left(\int_\Omega \nabla(\Delta v) : \nabla(\Delta v)\, dx \right)^{1/2}$$

(for matrices $A = (a_{ij})$ and $B = (b_{ij})$ of order n we put $A : B = \sum\limits_{i,j=1}^{n} a_{ij} b_{ij}$).

The operator A is a topological isomorphism between V^α and $V^{\alpha-2}$ for any $\alpha \in \mathbb{R}$. The operator $A\colon V^1 \to V^{-1}$ acts according to the formula

$$\langle Au, v \rangle = \int_\Omega \nabla u : \nabla v\, dx \qquad (u, v \in V^1).$$

We use standard notation for the spaces of integrable function with values in Banach spaces. Time derivatives are in the sense of distributions $\mathcal{D}(0,T;V^{-1})$.

We need the following Banach spaces in order to define weak solutions:

$$W_1[0,T] = \{v\colon v \in L_\infty(0,T;V^1),\ v' \in L_\infty(0,T;V^{-1})\}$$

with the norm

$$\|v\|_{W_1[0,T]} = \|v\|_{L_\infty(0,T;V^1)} + \|v'\|_{L_\infty(0,T;V^{-1})};$$

and

$$W_2[0,T] = \{v\colon v \in C([0,T];V^3),\ v' \in L_\infty(0,T;V^3)\}$$

with the norm

$$\|v\|_{W_2[0,T]} = \|v\|_{C([0,T];V^3)} + \|v'\|_{L_\infty(0,T;V^3)},$$

Also let $W_1^{\rm loc}(\mathbb{R}_+)$ be the class of functions $v\colon \mathbb{R}_+ \to V^1$ such that the restriction of v to any segment $[0,T]$ belongs to $W_1[0,T]$; likewise, let $W_2^{\rm loc}(\mathbb{R}_+)$ denote the class of functions $v \in C(\mathbb{R}_+, V^3)$ such that the restriction of v to any segment $[0,T]$ belongs to $W_2[0,T]$. These classes are needed for defining solutions on the nonnegative semiaxis.

The following compactness theorem is very important. Suppose that $X_0 \subset F \subset X_1$ be Banach spaces, where the first embedding is compact and X_0 is reflexive; further, let $T > 0$ and $1 \leqslant p_i \leqslant \infty$ $(i = 1,2)$. Consider the space

$$W(0,T;p_0,p_1;X_0,X_1) = \{u\colon u \in L_{p_0}(0,T;X_0),\ u' \in L_{p_1}(0,T;X_1)\}$$

(the time derivative is in the sense of distributions on $(0,T)$ with values in X_1); $W(0,T;p_1,p_2;X_0,X_1)$ is endowed with the norm

$$\|u\|_W = \|u\|_{L_{p_0}(0,T;X_0)} + \|u'\|_{L_{p_1}(0,T;X_1)}.$$

Theorem 2.1. *If $p_0 < \infty$, the following embedding is compact:*

$$W(0,T;p_0,p_1;X_0,X_1) \subset L_{p_0}(0,T;F);$$

If $p_0 = \infty$ and $p_1 > 1$, the following embedding is compact:

$$W(0,T;p_0,p_1;X_0,X_1) \subset C([0,T];F).$$

The proof can be found e. g. in [7].

3. THE PROBLEM DEFINITION AND MAIN RESULTS

Let the body force $f \in (L_2(\Omega))^n$ be fixed.

Definition 3.1. A function $v \in W_1[0,T]$ is called a *weak solution* of problem (1.1)–(1.4) on $[0,T]$ with the initial condition $a \in V^1$ if it satisfies the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega v(t)\cdot\varphi\,\mathrm{d}x + \varkappa\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega \nabla v(t) : \nabla\varphi\,\mathrm{d}x + \nu\int\limits_\Omega \nabla v(t) : \nabla\varphi\,\mathrm{d}x - \sum_{i,j=1}^n \int\limits_\Omega v_i(t)v_j(t)\frac{\partial\varphi_j}{\partial x_i}\mathrm{d}x - \varkappa\sum_{i,j,k=1}^n\int\limits_\Omega v_k(t)\frac{\partial v_i}{\partial x_j}(t)\frac{\partial^2\varphi_j}{\partial x_i\partial x_k}\mathrm{d}x - \varkappa\sum_{i,j,k=1}^n\int\limits_\Omega v_k(t)\frac{\partial v_j}{\partial x_i}(t)\frac{\partial^2\varphi_j}{\partial x_i\partial x_k}\mathrm{d}x = \int\limits_\Omega f\cdot\varphi\,\mathrm{d}x. \quad (3.1)$$

a. e. on $(0,T)$ for any $\varphi \in V^3$ and satisfies the initial condition

$$v(0) = a. \tag{3.2}$$

A function $v \in W_1^{\mathrm{loc}}(\mathbb{R}_+)$ is called a weak solution of problem (1.1)–(1.4) on $\mathbb{R}_+$ if for any $T > 0$ the function v is a weak solution of the problem on $[0,T]$.

Remark 3.1. If $v \in W_1[0,T]$, then $v(t) \in V \subset (L_4(\Omega))^n$ and $\frac{\partial v_i}{\partial x_j} \in L_2(\Omega)$ for almost all $t \in (0,1)$. For $\varphi \in V^3 \subset (H^3(\Omega))^n$ we have $\frac{\partial^2 \varphi_j}{\partial x_i \partial x_k} \in H^1(\Omega) \subset L_4(\Omega)$. Consequently, all the integrals in the left-hand side of (3.1) exist. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_\Omega v(t) \cdot \varphi \, \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}(v(t),\varphi)_0 = \langle v'(t), \varphi \rangle_{V^{-1} \times V^1},$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_\Omega \nabla v(t) : \nabla \varphi \, \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}(v(t),\varphi)_3 = \langle v'(t), \varphi \rangle_{V^{-3} \times V^3}$$

in the sense of scalar distributions.

Remark 3.2. By the following theorem

Theorem 3.1. *Let E and E_0 be Banach spaces, and let E be continuously embedded in E_0. If a function u belongs to $L_\infty(0,M;E)$ and is continuous as a function with values in E_0, then u is weakly continuous as a function with values in E.*

we have $W_1[0,T] \subset C_w([0,T];V^1)$. Thus the initial condition (3.2) is sensible for functions belonging to the class $W_1[0,T]$.

The identity (3.1) is derived from equations (1.1)–(1.3) in a standard way: under the assumption that a classical solution exists, multiply equation (1.1) by an aribtrary function $\varphi \in V^3$ and integrate by parts certain terms; since φ is solenoidal, the term $\operatorname{grad} p$ is eliminated.

The following existence theorem holds.

Theorem 3.2. *For any $a \in V^1$ the problem (1.1)–(1.4) has a solution on the semiaxis $\mathbb{R}_+$ that satisfies the inequality*

$$\|v(t)\|_1 + \|v'(t)\|_{-1} \leqslant R_0 \left(1 + \|a\|_1^2 e^{-\alpha t}\right) \quad \text{a. a. } t \geqslant 0 \tag{3.3}$$

where the constants $R_0 > 0$ and $\alpha > 0$ are independent of v.

Definition 3.2. A function $v \in W_1^{\mathrm{loc}}(\mathbb{R}_+) \cap L_\infty(\mathbb{R}_+;E)$ is called a *a trajectory* of problem (1.1)–(1.4) if it is a solution of this problem with some $a \in V^1$ and the following inequality holds:

$$\|v(t)\|_1 + \|v'(t)\|_{-1} \leqslant R_0 \left(1 + \|v\|^2_{L_\infty(\mathbb{R}_+,V^1)} e^{-\alpha t}\right) \quad \text{a. a. } t \geqslant 0. \tag{3.4}$$

The set of trajectories is called its *trajectory space* of the problem and is denoted by $\mathcal{H}^+$.

Remark 3.3. Weak solutions of problem (1.1)–(1.4) are weakly continuous in V^1, whence $\|a\|_{V^1} = \|v(0)\|_{V^1} \leqslant \|v\|_{L_\infty(\mathbb{R}_+,V^1)}$. Thus inequality (3.4) follows from inequality (3.3), and by Theorem 3.2 we see that any point $a \in V^1$ is the beginning of a trajectory.

Consider a number $\delta \in (0,1)$ and suppose that $f \in (L_2(\Omega))^n$.

These are the main results concerning the existence of attractors.

Theorem 3.3. *The trajectory space $\mathcal{H}^+$ has the minimal trajectory attractor $\mathcal{U}$. The attractor is bounded in $L_\infty(\mathbb{R}_+;V^1)$ and compact in $C(\mathbb{R}_+;V^{1-\delta})$; it attracts sets of trajectories bounded in $L_\infty(\mathbb{R}_+;V^1)$ with respect to the topology of $C(\mathbb{R}_+;V^{1-\delta})$.*

Theorem 3.4. *The trajectory space $\mathcal{H}^+$ has the global trajectory attractor $\mathcal{A}$. The attractor is bounded in V^1, compact in $V^{1-\delta}$; it attracts sets of trajectories bounded in $L_\infty(\mathbb{R}_+; V^1)$ with respect to the topology of $V^{1-\delta}$.*

These theorems are proved in Section after certain auxiliary results have been stated.

4. APPROXIMATING PROBLEM

Take $\varepsilon > 0$. Consider the following identity as an approximation of (3.1):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega v(t)\varphi\,\mathrm{d}x + \varepsilon \mathrm{e}^{-\alpha t}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega \nabla(\Delta v(t)) : \nabla(\Delta\varphi)\,\mathrm{d}x + \varkappa\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega \nabla v(t) : \nabla\varphi\,\mathrm{d}x + \nu\int\limits_\Omega \nabla v(t) : \nabla\varphi\,\mathrm{d}x$$
$$- \sum_{i,j=1}^n \int\limits_\Omega v_i(t)v_j(t)\frac{\partial\varphi_j}{\partial x_i}\mathrm{d}x - \varkappa\sum_{i,j,k=1}^n\int\limits_\Omega v_k(t)\frac{\partial v_i}{\partial x_j}(t)\frac{\partial^2\varphi_j}{\partial x_i\partial x_k}\mathrm{d}x$$
$$- \varkappa\sum_{i,j,k=1}^n\int\limits_\Omega v_k(t)\frac{\partial v_j}{\partial x_i}(t)\frac{\partial^2\varphi_j}{\partial x_i\partial x_k}\mathrm{d}x = \int\limits_\Omega f\varphi\,\mathrm{d}x \quad (\varphi\in V^3). \quad (4.1)$$

This identity transforms into (3.1) as $\varepsilon \to 0$.

In what follows we consider an operator equation generated by identity (4.1) rather than the identity itself. Consider the following operators:

$$N\colon V^3 \to V^{-3}, \quad \langle Nu,\varphi\rangle = \int\limits_\Omega \nabla(\Delta u) : \nabla(\Delta\varphi)\mathrm{d}x;$$
$$B_1\colon (L_4(\Omega))^n \to V^{-1},\ \langle B_1(u),\varphi\rangle = \sum_{i,j=1}^n\int\limits_\Omega u_iu_j\frac{\partial\varphi_j}{\partial x_i}\,\mathrm{d}x;$$
$$B_2\colon V^1 \to V^{-3}, \quad \langle B_2(u),\varphi\rangle = \sum_{i,j,k=1}^n\int\limits_\Omega u_k\frac{\partial u_i}{\partial x_j}\frac{\partial^2\varphi_j}{\partial x_i\partial x_k}\,\mathrm{d}x;$$
$$B_3\colon V^1 \to V^{-3}, \quad \langle B_2(u),\varphi\rangle = \sum_{i,j,k=1}^n\int\limits_\Omega u_k\frac{\partial u_j}{\partial x_i}\frac{\partial^2\varphi_j}{\partial x_i\partial x_k}\,\mathrm{d}x.$$

It will be convenient to have a notation for the exponential function. By definition, for any $\beta\in\mathbb{R}$ put
$$e_\beta(t) = \mathrm{e}^{\beta t}.$$

Identity (4.1) generates the following operator equation:

$$(I + \varepsilon e_{-\alpha}N + \varkappa A)v' + \nu Av - B_1(v) - \varkappa B_2(v) - \varkappa B_3(v) = \widetilde{f}, \quad (4.2)$$

where $\widetilde{f} = \pi f \in V^0 \subset V^{-1}$, π is the Leray projector and thus

$$\langle\widetilde{f},\varphi\rangle_{V^{-1}.V^1} = \int\limits_\Omega \widetilde{f}\cdot\varphi\,\mathrm{d}x = \int\limits_\Omega f\cdot\varphi\,\mathrm{d}x \quad (\varphi\in V^3),$$

A function $v\in W_2[0,T]$ is called a *solution* of equation (4.2) on $[0,T]$ if it yields a true equality in $L_1(0,T;V^{-3})$ when substitued into (4.2). A function $v\in W_2^{\mathrm{loc}}(\mathbb{R}_+)$ is called a solution of (4.2) on $\mathbb{R}_+$ if it is a solution of (4.2) on each finite segment $[0,T]$.

Since we look for solutions of (4.2) in the class W_2, which consists of continuous functions with values in V^3, it is sensible to provide an initial condition of the form

$$v(0) = b \tag{4.3}$$

with $b \in V^3$.

The existence theorem for solutions of the approximating problem is considered in Section .

5. A PRIORI ESTIMATES

Put

$$\alpha = \frac{\nu}{K_0^2 + \varkappa}.$$

We shall use this notation throughout the entire section.

We use topological methods in order to prove that problem (4.2), (4.3) has solutions. Given $\varepsilon > 0$, consider the family of problems depending on the parameter $\lambda \in [0, 1]$:

$$(I + \varepsilon e_{-\lambda\alpha} N + \varkappa A)v' + \lambda \left(\nu A v - B_1(v) - \varkappa B_2(v) - \varkappa B_3(v)\right) = \lambda \widetilde{f}, \tag{5.1}$$

$$v(0) = \lambda b. \tag{5.2}$$

The notion of solution has the same sense for (5.1) as for (4.2).

In problem (5.1), (5.2) λ is the parameter of a nonlinear deformation. If $\lambda = 1$, problem (5.1), (5.2) yields the original problem (4.2), (4.3). If $\lambda = 0$, problem (5.1), (5.2) is reduced to a simpler linear problem whose solvability can be established by standard methods. The deformation is considered in more detail below.

Let $v \in W_2[0, T]$ be a solution of (5.1) on $[0, T]$ for certain $\lambda \in [0, 1]$. It can be proved that in this case the left-hand side of (5.1) belongs to $L_\infty(0, T; V^{-3})$, and *a fortiori* the equation holds in $L_2(0, T; V^{-3})$. Apply both sides to $v(t)$ and observe that

$$\langle v'(t), v(t)\rangle = (v'(t), v(t))_0 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_0^2,$$

$$\langle \mathrm{e}^{-\lambda\alpha t} N v'(t), v(t)\rangle = \mathrm{e}^{-\lambda\alpha t}(v'(t), v(t))_3 = \mathrm{e}^{-\lambda\alpha t}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_3^2,$$

$$\langle A v'(t), v(t)\rangle = (v'(t), v(t))_1 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_1^2,$$

moreover, it is known that $\langle B_i(v(t)), v(t)\rangle = 0$ $(i = 1, 2, 3)$ [8]. Thus we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_0^2 + \frac{\varepsilon}{2}\mathrm{e}^{-\lambda\alpha t}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_3^2 + \frac{\varkappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_1^2 + \lambda\nu\|v(t)\|_1^2 = \int\limits_\Omega \widetilde{f} \cdot v(t)\,\mathrm{d}t. \tag{5.3}$$

Now we demonstrate how a dissipative estimate with a decaying exponential can be derived from (5.3). We estimate the right-hand side of the latter equation using the Cauchy inequality:

$$\int\limits_\Omega \widetilde{f} \cdot v(t)\,\mathrm{d}t = \langle \widetilde{f}, v(t)\rangle_{V^{-1}\times V^1} \leqslant \|\widetilde{f}\|_{-1}\|v(t)\|_1 \leqslant \frac{1}{2\nu}\|\widetilde{f}\|_{-1}^2 + \frac{\nu}{2}\|v(t)\|_1^2.$$

Combining this with (5.3), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_0^2 + \varkappa\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_1^2 + \varepsilon\mathrm{e}^{-\lambda\alpha t}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_3^2 + \lambda\nu\|v(t)\|_1^2 \leqslant \frac{\lambda}{\nu}\|\widetilde{f}\|_{-1}^2. \tag{5.4}$$

Consider an auxiliary norm on V^1 defined by the formula $\|u\|^2 = \|u\|_0^2 + \varkappa\|u\|_1^2$. This norm is equivalent to $\|\cdot\|_1$. We have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_0^2 + \varkappa\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_1^2 = \frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2;\ \nu\|v(t)\|_1^2 \geqslant \frac{\nu}{K_0+\varkappa}\|v(t)\|^2 = \alpha\|v(t)\|^2.$$

Thus it follows from (5.4) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2 + \varepsilon e^{-\lambda\alpha t}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_3^2 + \lambda\alpha\|v(t)\|^2 \leqslant \frac{\lambda}{\nu}\|\widetilde{f}\|_{-1}^2.$$

Substitute $v(t) = \bar{v}(t)\exp(-\lambda\alpha t/2)$ in the first and the third terms in the left-hand side of the last inequality. We get

$$-\lambda\alpha \mathrm{e}^{-\lambda\alpha t}\|\bar{v}(t)\|^2 + \mathrm{e}^{-\lambda\alpha t}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{v}(t)\|^2 + \varepsilon\mathrm{e}^{-\lambda\alpha t}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_3^2 + \lambda\alpha\mathrm{e}^{-\lambda\alpha t}\|\bar{v}(t)\|^2 \leqslant \frac{\lambda}{\nu}\|\widetilde{f}\|_{-1}^2.$$

Multiplying both sides by $\exp(\lambda\alpha t)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\bar{v}(t)\|^2 + \varepsilon\|v(t)\|_3^2) \leqslant \frac{\lambda}{\nu}\|\widetilde{f}\|_{-1}^2\mathrm{e}^{\lambda\alpha t}. \tag{5.5}$$

Integrating the last inequality, we have

$$\|\bar{v}(t)\|^2 + \varepsilon\|v(t)\|_3^2 \leqslant \|v(0)\|^2 + \varepsilon\|v(0)\|_3^2 + \frac{1}{\alpha\nu}\|\widetilde{f}\|_{-1}^2\left(\mathrm{e}^{\lambda\alpha t} - 1\right),$$

for all t (this is true both for $\lambda > 0$ and for $\lambda = 0$). Now multiply both parts of the last inequality by $\exp(-\lambda\alpha t)$, whence we obtain

$$\|v(t)\|^2 + \varepsilon\mathrm{e}^{-\lambda\alpha t}\|v(t)\|_3^2 \leqslant \frac{1}{\alpha\nu}\|\widetilde{f}\|_{-1}^2 + \left(\|v(0)\|^2 + \varepsilon\|v(0)\|_3^2\right)\mathrm{e}^{-\lambda\alpha t}.$$

Since the norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent, it follows from the last equality that

$$\|v(t)\|_1^2 + \varepsilon\mathrm{e}^{-\alpha t}\|v(t)\|_3^2 \leqslant C\left(1 + \left(\|v(0)\|_1^2 + \varepsilon\|v(0)\|_3^2\right)\mathrm{e}^{-\lambda\alpha t}\right) \tag{5.6}$$

with a constant C independent of λ, ε, and v.

Using (5.1) it is possible to estimate the derivative v' in terms of v. Combining the estimate obtained in this way with (5.6), we obtain

$$\|v(t)\|_1 + \sqrt{\varepsilon}\mathrm{e}^{-\alpha t/2}\|v(t)\|_3 + \|v'(t)\|_{-1} + \varepsilon\mathrm{e}^{-\alpha t}\|v'(t)\|_3 \leqslant R_1\left(1 + \left(\|v(0)\|_1^2 + \varepsilon\|v(0)\|_3^2\right)\mathrm{e}^{-\lambda\alpha t}\right). \tag{5.7}$$

with a constant R_1 that does not depend on ε, λ, and v.

6. EXISTENCE OF SOLUTIONS

Now we state the main existence theorem for the approximating problem.

Theorem 6.1. *For any $b \in V^3$ problem* (4.2), (4.3) *has a solution on the semiaxis $\mathbb{R}_+$. Any solution of this problem satisfies*

$$\|v(t)\|_1 + \sqrt{\varepsilon}\mathrm{e}^{-\alpha t/2}\|v(t)\|_3 + \|v'(t)\|_{-1} + \varepsilon\mathrm{e}^{-\alpha t}\|v'(t)\|_3 \leqslant R_1\left(1 + \left(\|v(0)\|_1^2 + \varepsilon\|v(0)\|_3^2\right)\mathrm{e}^{-\lambda\alpha t}\right) \tag{6.1}$$

a. e. on $\mathbb{R}_+$ with a constant R_1 independent of ε, λ, and v.

The proof of Theorem 6.1 involves two steps. First we prove the solvability on a finite segment $[0,T]$ with an arbitrary $T>0$ and then we prove that there exists a solution on $\mathbb{R}_+$.

Step I. Let $T>0$. Let us prove that problem (4.2), (4.3) has a solution on $[0,T]$.

Consider the following family of operators depending on $\lambda \in [0,1]$:

$$L_\lambda\colon W_2[0,T] \to L_\infty(0,T;V^{-3}) \times V^3,$$
$$L_\lambda(v) = \big((I + \varepsilon e_{-\lambda\alpha} N + \varkappa A)v', v(0)\big)\,;$$

and the operator

$$K\colon W_2[0,T] \to L_\infty(0,T;V^{-3}) \times V^3,$$
$$K(v) = (\nu A v - B_1(v) - \varkappa B_2(v) - \varkappa B_2(v), 0)\,.$$

It can be proved [4] that for any $\lambda \in [0,1]$ the linear operator

$$L_\lambda\colon W_2[0,T] \to L_\infty(0,T;V^{-3}) \times V^3$$

is bounded and invertible, and the inverse operator depends continuously on λ in the operator norm. (The proof is based on the fact that the inverse operator can be expressed explicitly) [3]. It can be proved that $K\colon W_2[0,T] \to L_\infty(0,T;V^{-3}) \times V^3$ is compact.

Consider the family of equations dependent on $\lambda \in [0,1]$.

$$L_\lambda v + \lambda K(v) = \lambda(f,b), \tag{6.2}$$

For any λ equation (6.2) is equivalent to problem (5.1), (5.2). In particular, equation (6.2) with $\lambda = 1$ corresponds to (4.2), (4.3). Note that it follows from (5.7) that solutions of (6.2) (if they exist) satisfy the following *a priori* estimate:

$$\|v\|_{C([0,T];V^3)} + \varepsilon \mathrm{e}^{-\alpha T} \|v'\|_{L_\infty(0,T;V^3)} \leqslant C, \tag{6.3}$$

where C does not depend on λ (but generally speaking, it can depend on other parameters of the equation). Indeed, it follows from (5.7) that for a. a. $t \in [0,T]$ the norms $\|v(t)\|_3$ and $\mathrm{e}^{-\alpha T}\|v'(t)\|_3$ do not exceed

$$R_1(1 + (\lambda^2\|a\|_1^2 + \varepsilon\lambda^2\|a\|_3^2))\mathrm{e}^{-\lambda\alpha t} \leqslant R_1(1 + (\|a\|_1^2 + \varepsilon\|a\|_3^2)),$$

and the right-hand part of the last inequality does not depend on t and λ. Also it follows from 6.3 that solutions of (6.2) satisfy

$$\|v\|_{W_2[0,T]} \leqslant R, \tag{6.4}$$

where R does not depend on λ.

Apply L_λ^{-1} to both sides of (6.2) and write the equation thus obtained in the form

$$v - \lambda L_\lambda^{-1}((f,a) - K(v)) = 0. \tag{6.5}$$

The mapping $\Phi(\lambda, v) = \lambda L_\lambda^{-1}((f,a) - K(v))$ is continuous with respect to (λ, v), so it is a deformation between vector fields $\Phi_1 v = v - L_1^{-1}((f,a) - K(v))$ and $\Phi_0 v = v$. It can be proved that $\Phi(\lambda, v)$ regarded as a function of v is uniformly continuous with respect to λ. Moreover, it follows from (6.4) that $\Phi(\lambda, v)$ does not vanish on the boundary of the ball B_{R+1}. Hence $\Phi(\lambda, v)$ is a homotopy between $\Phi_1 v$ and $\Phi_2 v$ on B_{R+1}.

Since the deformation $\Phi(\lambda, v)$ is nondegenerate on the boundary of B_{R+1}, the Leray–Schauder degree of the completely continuous vector fields $\Phi_1 v$ and $\Phi_0 v$ on B_{R+1} is well defined. By the homotopic invariance of the Leray–Schauder degree we have

$$\deg_{\mathrm{LS}}(\mathrm{id} - L_1^{-1}((f,a) - K(\cdot)), B_{R+1}, 0) = \deg_{\mathrm{LS}}(\mathrm{id}, B_{R+1}, 0) = 1.$$

Since the field $\mathrm{id} - L_1^{-1}((f,a) - K(\cdot))$ has non-zero degree, there exists a solution $v \in W_2[0,T]$ of the operator equation

$$v - L_1^{-1}((f,a) - K(v)) = 0.$$

This equation is equivalent to equation (6.2) with $\lambda = 1$, and the latter equation is in turn equivalent to problem (4.2), (3.2). We have thus proved that problem (4.2), (4.3) has a solution on $[0,T]$.

Step II. Let v_m be a solution of problem (4.2), (4.3) on $[0,m]$ $(m = 1, 2, \ldots)$. Consider the extension of the functions v_m to $\mathbb{R}_+$ defined by the formula

$$\widehat{v}_m(t) = \begin{cases} v(t), & 0 \leqslant t \leqslant m, \\ v(m), & t \geqslant m. \end{cases}$$

It is obvious that the functions $\widetilde{v}_m$ belong to $W_2^{\mathrm{loc}}(\mathbb{R}_+)$.

Suppose that $0 < \delta < 1$. Take an arbitrary $T > 0$. All but finitely many terms of the sequence $\{\widehat{v}_m\}$ are solutions of (4.2), (4.3) on $[0,T]$. Since the functions $\widehat{v}_m$ take the same value b at 0, by Theorem 6.1 it follows that they satisfy the estimate

$$\|\widehat{v}_m\|_{L_\infty(0,T;V^1)} + \|\widehat{v}_m\|_{L_\infty(0,T;V^3)} + \|\widehat{v}'_m\|_{L_\infty(0,T;V^3)} + \|\widehat{v}'_m\|_{L_\infty(0,T;V^{-1})} \leqslant C(\varepsilon, T), \tag{6.6}$$

where $C(\varepsilon, T)$ does not depend on m. Thus the sequence $\{\widehat{v}_m\}$ is bounded in $L_\infty(0,T;V^1)$ and the sequence of derivatives $\{\widehat{v}'_m\}$ is bounded with respect to the norm of $L_\infty(0,T;V^{-1})$. By Theorem 2.1 we have that the sequence $\{\widehat{v}_m\}$ is precompact in $C([0,T];V^{1-\delta})$. Since this is true for arbitrary T, the sequence is precompact in $C(\mathbb{R}_+;V^{1-\delta})$.

Thus the sequence $\{\widehat{v}_m\}$ has a subsequence $\{\widehat{v}_{m_k}\}$ that converges to some function v_* in the space $C(\mathbb{R}_+, V^{1-\delta})$. It can be proved [4] that this limit function is the the sought for solution of problem (4.2), (4.3) on $\mathbb{R}_+$.

Proof of Theorem 3.2. Since V^3 is dense in V^1, there exists a sequence $\{b_m\}$ in V^3 such that $\|b_m - a\|_1 \to 0$. Suppose the sequence $\{\varepsilon_m\}$ tends to zero and

$$\varepsilon_m \|b_m\|_3^2 \leqslant 1. \tag{6.7}$$

One can put e. g.

$$\varepsilon_m = \frac{1}{m \max\{\|b_m\|_3^2, 1\}}$$

to obtain such a sequence.

Substitute ε_m for ε in (4.2) and consider the initial condition

$$v_m(0) = b_m$$

for this equation. By Theorem 6.1 this initial value problem has a solution v_m on $\mathbb{R}_+$. Inequalities (6.1) and (6.7) yield the following estimate:

$$\|v_m(t)\|_1 + \|v'_m(t)\|_{-1} + \varepsilon_m \mathrm{e}^{-\alpha t}\|v'_m(t)\|_3 \leqslant R_1 \left(1 + \left(\|a_m\|_1^2 + 1\right) \mathrm{e}^{-\alpha t}\right), \tag{6.8}$$

a. e. on $\mathbb{R}_+$. More precisely, for each m the last inequality holds for all $t \in \mathbb{R}_+ \setminus Q_m$, where Q_m is a set of zero measure. Hence for any $t \in \mathbb{R}_+ \setminus Q$, where $Q = \cup_m Q_m$ is a set of zero measure, inequality (6.8) holds for *all* m.

Suppose that $0 < \delta \leqslant 1$. According to (6.8) we have that for any $T > 0$ the sequence $\{v_m\}$ is bounded in $L_\infty(0,T;V^1)$ and the sequence $\{v'_m\}$ is bounded in $L_\infty(0,T;V^{-1})$. By Theorem 2.1 it follows that the sequence $\{v_m\}$ is compact in $C([0,T];V^{1-\delta})$. Since T is arbitrary, it follows that the latter sequence is precompact in $C(\mathbb{R}_+, V^{1-\delta})$ and thus has a subsequence $\{v_{m_k}\}$ converging in $C(\mathbb{R}_+, V^{1-\delta})$ to a function v_*. It is proved in [4] (cf. [3], [8]) that v_* is a solution of problem (3.1), (3.2).

Now we demonstrate (3.3). Discarding certain nonnegative terms in the left-hand side of (6.8) we obtain

$$\|v_{m_k}(t)\|_1 \leqslant R_1 \left(1 + \left(\|a_m\|_1^2 + 1\right) \mathrm{e}^{-\alpha t}\right). \tag{6.9}$$

Given k, this inequality holds for any t belonging to a subset of $\mathbb{R}_+$ of full measure that does not depend on k. Take such a t. First observe that $v_{m_k}(t) \to v_*(t)$ in $V^{1-\delta}$, since the convergence in $C(\mathbb{R}_+, V^{1-\delta})$ implies pointwise convergence. Further, it follows from (6.9) that the sequence $\{v_{m_k}(t)\}$ is bounded in V^1. Consequently, it has a subsequence $\widetilde{v}_\mu(t)$ that converges to $v_*(t)$ weakly in V^1. Therefore

$$\|v_*(t)\|_1 \leqslant \varliminf_{\mu\to\infty} \|\widetilde{v}_\mu(t)\|_1 \leqslant R_1 \left(1 + \left(\|a\|_1^2 + 1\right) \mathrm{e}^{-\alpha t}\right).$$

Thus for a. a. $t \in \mathbb{R}_+$ we have

$$\|v_*(t)\|_1 \leqslant R_1 \left(1 + \left(\|a\|_1^2 + 1\right) \mathrm{e}^{-\alpha t}\right). \tag{6.10}$$

Moreover, one can use (3.1) and estimate v_*' in terms of v. Combining such an estimate with (6.10), we get (3.3). □

7. TRAJECTORY SPACE AND ATTRACTORS

In this subsection we fix a number $\delta \in (0, 1)$.

Consider $E = V^1$ and $E_0 = V^{1-\delta}$ as the pair of Banach spaces needed to introduce a trajectory space. This choice is justified by the fact that V^1 is reflexive and is continuously embedded in $V^{1-\delta}$.

By Remark 3.3 the trajectory space introduced by Definition 3.2 is nonempty. Thus it suffices to check the inclusion

$$\mathcal{H}^+ \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E).$$

so as to make sure that the trajectory space is well defined.

The inclusion $\mathcal{H}^+ \subset L_\infty(\mathbb{R}_+; E)$ directly follows from the definition of the trajectory space. We use Theorem 2.1 in order to prove that the trajectories are continuous. Consider three spaces $V^1 \subset V^{1-\delta} \subset V^{-1}$. Let v be an arbitrary trajectory. It follows from (3.4) that for any segment $[0, T]$ we have $v \in L_\infty(0, T; V^1)$ and $v' \in L_\infty(0, T; V^{-1})$. Hence by Theorem 2.1 we obtain that v belongs to $C([0, T]; V^{1-\delta})$. This is true for any T, so $v \in C(\mathbb{R}_+; V^{1-\delta})$, q. e. d.

Let $\widetilde{R} \geqslant 4R_0$. Consider the set

$$\widetilde{P} = \{v \in C(\mathbb{R}_+; V^{1-\delta}) \cap L_\infty(\mathbb{R}_+; V^1) \colon v' \in L_\infty(\mathbb{R}_+, V^{-1}), \\ \|v\|_{L_\infty(\mathbb{R}_+; V^1)} + \|v'\|_{L_\infty(\mathbb{R}_+; V^{-1})} \leqslant \widetilde{R}\}. \tag{7.1}$$

Let us establish several properties of this set.

Lemma 7.1. *The set $\widetilde{P}$ is bounded in $L_\infty(\mathbb{R}_+; V^1)$, compact in $C(\mathbb{R}_+; V^{1-\delta})$, and the following inclusion holds:*

$$\mathrm{T}(h)\widetilde{P} \subset \widetilde{P} \quad (h \geqslant 0). \tag{7.2}$$

Доказательство. It follows from the definition of $\widetilde{P}$ that it is bounded in $L_\infty(\mathbb{R}_+; V^1)$.

It is not hard to prove that the set $\widetilde{P}$ is precompact in $C(\mathbb{R}_+; V^{1-\delta})$. Indeed, take $T > 0$. It follows easily from the construction that $\widetilde{P}$ is bounded in $L_\infty(0, T; V^1)$ and the set $\{v' \colon v \in \widetilde{P}\}$ is bounded in $L_\infty(0,T;V^{-1})$. By Theorem 2.1 the set $\widetilde{P}$ is precompact in $C([0,T];V^{1-\delta})$. Since T is arbitrary, $\widetilde{P}$ is precompact in $C(\mathbb{R}_+;V^{1-\delta})$.

Now let us show that $\widetilde{P}$ is closed and therefore compact in $C(\mathbb{R}_+;V^{1-\delta})$. Suppose that the sequence $\{v_m\}\subset\widetilde{P}$ converges to v_0 in $C(\mathbb{R}_+;V^{1-\delta})$. This sequence is bounded in $L_\infty(\mathbb{R}_+;V^1)$, so it converges to its limit function $*$-weakly in $L_\infty(\mathbb{R}_+;V^1)$. Moreover, the sequence of derivatives $\{v'_m\}$ converges to v'_0 in the sense of distributions and also $*$-weakly in $L_\infty(\mathbb{R}_+;V^{-1})$, since it is bounded in $L_\infty(\mathbb{R}_+;V^{-1})$. Therefore

$$\|v_0\|_{L_\infty(\mathbb{R}_+;V^1)}+\|v'_0\|_{L_\infty(\mathbb{R}_+;V^{-1})} \leqslant \varliminf_{m\to\infty}\|v_m\|_{L_\infty(\mathbb{R}_+;V^1)}+\varliminf_{m\to\infty}\|v'_m\|_{L_\infty(\mathbb{R}_+;V^{-1})} \leqslant \varliminf_{m\to\infty}\big(\|v_m\|_{L_\infty(\mathbb{R}_+;V^1)}+\|v'_m\|_{L_\infty(\mathbb{R}_+;V^{-1})}\big)\leqslant\widetilde{R}.$$

This proves that $\widetilde{P}$ contains the limit function v_0. So $\widetilde{P}$ is closed.

Finally, let us prove the inclusion (7.2). Take $h\geqslant 0$. For any $v\in\widetilde{P}$ we have

$$\|\mathrm{T}(h)v\|_{L_\infty(\mathbb{R}_+;V^1)}+\|\mathrm{T}(h)v\|_{L_\infty(\mathbb{R}_+;V^{-1})} \leqslant\|v\|_{L_\infty(\mathbb{R}_+;V^1)}+\|v'\|_{L_\infty(\mathbb{R}_+;V^{-1})}\leqslant\widetilde{R},$$

whence $\mathrm{T}(h)v\in\widetilde{P}$, q. e. d. □

Proof of Theorem 3.3. Let us prove that $\widetilde{P}$ is a semi-attractor of $\mathcal{H}^+$. By Lemma 7.1 we have that $\widetilde{P}$ satisfies conditions (i) and (ii) of the trajectory semi-attractor definition:

Definition 7.1. A nonempty set $P\subset C(\mathbb{R}_+;E_0)\cap L_\infty(\mathbb{R}_+;E)$ is called a *trajectory semi-attractor* of the trajectory space $\mathcal{H}^+$, if the following conditions hold:

(i) P is compact in $C(\mathbb{R}_+;E_0)$ and bounded in $L_\infty(\mathbb{R}_+;E)$;

(ii) the inclusion $\mathrm{T}(t)P\subset P$ holds for all $t\geqslant 0$;

(iii) P is an *attracting set*, i. e. for any nonempty set $B\subset\mathcal{H}^+$ bounded with respect to the norm of $L_\infty(\mathbb{R}_+;E)$ we have
$$\lim_{t\to\infty}\sup_{u\in B}\inf_{v\in P}\|\mathrm{T}(t)u-v\|_{C(\mathbb{R}_+;E_0)}=0 \tag{7.3}$$
or equivalently
$$\lim_{t\to\infty}\sup_{u\in B}\inf_{v\in P}\|\mathrm{T}(t)u-v\|_{C([0,M];E_0)}=0\quad\forall\, M>0. \tag{7.4}$$

Let us prove that $\widetilde{P}$ is absorbing. Consider an arbitrary set $B\subset\mathcal{H}^+$ bounded in $L_\infty(\mathbb{R}_+;V^1)$; to be definite, assume that $\|v\|_{L_\infty(\mathbb{R}_+;V^1)}\leqslant R$ for any $v\in B$. Take $h_0\geqslant 0$ such that $R^2\mathrm{e}^{-\alpha h_0}\leqslant 1$. Let v be an arbitrary function belonging to B. Since v satisfies inequality (3.4), for all $h\geqslant h_0$ we have

$$\|\mathrm{T}(h)v(t)\|_1+\|\mathrm{T}(h)v'(t)\|_{-1}=\|v(t+h)\|_1+\|v'(t+h)\|_{-1}\leqslant \leqslant R_0(1+R^2\mathrm{e}^{-\alpha(t+h)})\leqslant R_0(1+R^2\mathrm{e}^{-\alpha h_0})\leqslant 2R_0.$$

Hence $\|\mathrm{T}(h)v\|_{L_\infty(\mathbb{R}_+;V^1)}\leqslant 2R_0$, $\|\mathrm{T}(h)v'\|_{L_\infty(\mathbb{R}_+;V^{-1})}\leqslant 2R_0$, and therefore

$$\|\mathrm{T}(h)v\|_{L_\infty(\mathbb{R}_+;V^1)}+\|\mathrm{T}(h)v'\|_{L_\infty(\mathbb{R}_+;V^{-1})}\leqslant 4R_0\leqslant\widetilde{R}.$$

Thus $\mathrm{T}(h)v\in\widetilde{P}$. Since v is arbitrary, we have $\mathrm{T}(h)B\subset\widetilde{P}$ for all $h\geqslant h_0$. Consequently $\widetilde{P}$ is absorbing.

We have proved that $\widetilde{P}$ is a semi-attractor of $\mathcal{H}^+$. By the following theorem

Theorem 7.1. *Suppose the trajectory space $\mathcal{H}^+$ has a trajectory semi-attractor P. Then $\mathcal{H}^+$ has the minimal trajectory attractor $\mathcal{U}$, the following relations hold:*

$$\Pi_+\mathcal{K}(\mathcal{H}^+) \subset \mathcal{U} = \Pi_+\mathcal{K}(\mathcal{U}) \subset \Pi_+\mathcal{K}(P) \subset P, \tag{7.5}$$

and the kernel $\mathcal{K}(\mathcal{H}^+)$ is compact in $C(\mathbb{R}; E_0)$ and bounded with respect to the norm of $L_\infty(\mathbb{R}; E)$.

the trajectory space $\mathcal{H}^+$ has the minimal trajectory attractor. □

Proof of Theorem 3.4. According to the following theorem

Theorem 7.2. *Suppose the trajectory space $\mathcal{H}^+$ has the minimal trajectory attractor $\mathcal{H}^+$. Then the global attractor $\mathcal{A}$ of $\mathcal{H}^+$ exists and the following relations hold:*

$$\mathcal{A} = \mathcal{U}(t), \quad t \geqslant 0; \tag{7.6}$$

$$\mathcal{K}(\mathcal{H}^+)(t) \subset \mathcal{A} = \mathcal{K}(\mathcal{U})(t), \quad t \in \mathbb{R}. \tag{7.7}$$

the global attractor of a trajectory space exists if the trajectory space has the minimal trajectory attractor. Theorem 3.3 implies that the trajectory space $\mathcal{H}^+$ satisfies this requirement. □

8. SUFFICIENT CONDITIONS OF CONVERGENCE

This section deals with the convergence of attractors. Suppose we are given a family of trajectory spaces $\mathcal{H}_\lambda^+ \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$ depending on a parameter λ that ranges over a metric space Λ (as before, we assume that $E \subset E_0$ are Banach spaces and that E is reflexive). Further, suppose that each trajectory space $\mathcal{H}_\lambda^+$ has the minimal trajectory attractor $\mathcal{U}_\lambda$ and the global attractor $\mathcal{A}_\lambda$ (the latter is the section of the former, according to the general theory). We want to establish sufficient conditions for $\mathcal{U}_\lambda$ to tend to $\mathcal{U}_{\lambda_0}$ and $\mathcal{A}_\lambda$ to tend to $\mathcal{A}_{\lambda_0}$ as $\lambda \to \lambda_0$.

From the point of view of applications $\mathcal{U}_{\lambda_0}$ and $\mathcal{A}_{\lambda_0}$ are the attractors of an unperturbed problem, while $\mathcal{U}_\lambda$ and $\mathcal{A}_\lambda$ for $\lambda \neq \lambda_0$ are the attractors of the approximating problem corresponding to possible values of the approximation parameter.

We consider the convergence in the sense of Hausdorff semi-distance in corresponding metric spaces. Recall that the *Hausdorff semi-distance* from a set A to a set B in a metric space (X, d) is given by

$$h_X(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b) \equiv \sup_{a \in A} \operatorname{dist}_X(a, B),$$

where $\operatorname{dist}_X(a, B)$ denotes the distance between a point a and a set B. In our case we consider the Hausdorff semi-distances $h_{C(\mathbb{R}_+; E_0)}$ in the space $C(\mathbb{R}_+; E_0)$ and h_{E_0} in the space E_0.

As before, suppose that E and E_0 are Banach spaces, E is continuously embedded in E_0, and E is reflexive.

The following proposition offers a sufficient condition for minimal trajectory attractors to converge in the sense of the Hausdorff semi-distance.

Proposition 8.1. *Suppose that a trajectory space*

$$\mathcal{H}_\lambda^+ \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E), \tag{8.1}$$

is assigned to every λ belonging to a metric space Λ. Suppose that each space $\mathcal{H}_\lambda^+$ has the minimal trajectory attractor $\mathcal{A}_\lambda$, which is contained in a set

$$P \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E),$$

P is precompact in $C(\mathbb{R}_+; E_0)$ and independent of λ. Moreover, suppose that the following condition holds:

(C) if $\lambda_m \to \lambda_0$, $u_m \in \mathcal{U}_{\lambda_m}$, and $u_m \to u_0$ in $C(\mathbb{R}_+; E_0)$, then $u_0 \in \mathcal{U}_{\lambda_0}$.

Then

$$h_{C(\mathbb{R}_+;E_0)}(\mathcal{U}_\lambda, \mathcal{U}_{\lambda_0}) = \sup_{u \in \mathcal{U}_\lambda} \inf_{v \in \mathcal{U}_{\lambda_0}} \|u - v\|_{C(\mathbb{R}_+;E_0)} \to 0 \qquad (\lambda \to \lambda_0). \tag{8.2}$$

Доказательство. Assume that (8.2) does not hold. This means that there exist a $\delta > 0$ and sequences $\{\lambda_m\}$ and $\{u_m\}$ such that $\lambda_m \to \lambda_0$, $u_m \in \mathcal{U}_{\lambda_m}$ and

$$\mathrm{dist}_{C(\mathbb{R};E_0)}(u_m, \mathcal{U}_{\lambda_0}) \geqslant \delta. \tag{8.3}$$

Since the sequence $\{u_m\}$ is contained in the precompact set P we can assume without loss of generality that it converges in $C(\mathbb{R}_+; E_0)$ to a limit function u_0. Passing to the limit in inequality (8.3), we obtain

$$\mathrm{dist}_{C(\mathbb{R};E_0)}(u_0, \mathcal{U}_{\lambda_0}) \geqslant \delta.$$

However, by condition (C) we have $u_0 \in \mathcal{U}_{\lambda_0}$, which contradicts the last inequality. This contradiction concludes the proof. □

The global attractor is a section of the minimal trajectory attractor. Hence it is not hard to prove that the convergence of minimal trajectory attractors implies the convergence of the global ones. Specifically, we have the following assertion.

Proposition 8.2. *Suppose that a trajectory space*

$$\mathcal{H}^+_\lambda \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E),$$

is assigned to every λ belonging to a metric space Λ. Suppose that each space $\mathcal{H}^+_\lambda$ λ possesses the minimal trajectory attractor $\mathcal{U}_\lambda$ and the global attractor $\mathcal{A}_\lambda = \mathcal{U}_\lambda(0)$, and suppose that (8.2) *holds. Then*

$$h_{E_0}(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) = \sup_{u \in \mathcal{A}_\lambda} \inf_{v \in \mathcal{A}_{\lambda_0}} \|u - v\|_{E_0} \to 0 \qquad (\lambda \to \lambda_0). \tag{8.4}$$

Доказательство. Condition (8.2) is equivalent to the following relation:

$$\lim_{\lambda \to \lambda_0} \sup_{u \in \mathcal{U}_\lambda} \inf_{v \in \mathcal{U}_{\lambda_0}} \|u - v\|_{C([0,M];E_0)} = 0 \qquad \forall M \geqslant 0.$$

In particular, letting $M = 0$ we get (8.4), since $\mathcal{A}_\lambda = \mathcal{U}_\lambda(0)$. □

Proposition (8.1) is not efficient enough, since it requires checking condition (C), which involves not trajectories, but whatever functions belonging to the minimal trajectory attractor. In what follows we consider an effective method of checking (C) in a certain class of trajectory spaces whose attractors can be represented as ω-limit sets.

From now on brackets denote the closure in $C(\mathbb{R}_+; E_0)$.

Definition 8.1. The *ω-limit set* for a set

$$P \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E), \tag{8.5}$$

bounded in $L_\infty(\mathbb{R}_+; E)$ is the set

$$\omega(P) = \bigcap_{t \geqslant 0} \left[\bigcup_{s \geqslant t} \mathrm{T}(s) P \right]. \tag{8.6}$$

Proposition 8.3. *Suppose that the set* (8.5) *is bounded in* $L_\infty(\mathbb{R}_+; E)$. *Then a function* u *belongs to* $\omega(P)$ *if and only if there exist sequences* $\{u_m\} \subset P$ *and* $\{t_m\} \subset \mathbb{R}_+$ *such that* $t_m \to \infty$ *and* $\mathrm{T}(t_m)u_m \to u$ *in* $C(\mathbb{R}_+; E_0)$.

Доказательство. Necessity. Take a positive integer m. It follows from (8.6) that

$$\omega(P) \subset \left[\bigcup_{s \geqslant m} \mathrm{T}(s)P\right],$$

whence there exists a function $u_m \in P$ and a number $t_m \geqslant m$ such that

$$\|\mathrm{T}(t_m)u_m - u\|_{C(\mathbb{R}_+;E_0)} < \frac{1}{m}.$$

The last inequality implies that $\mathrm{T}(t_m)u_m \to u$ in $C(\mathbb{R}_+; E_0)$, whence the sequences $\{u_m\}$ and $\{t_m\}$ are suitable.

Sufficiency. Take $t \geqslant 0$. We have $t_m \geqslant t$ whenever m is great enough. Hence all but finitely many terms of the sequence $\{\mathrm{T}(t_m)u_m\}$ lie in the set $\bigcup_{s\geqslant t} \mathrm{T}(s)P$. Consequently, the limit u of the sequence belongs to the closure $\left[\bigcup_{s\geqslant t} \mathrm{T}(s)P\right]$. This is true for any $t \geqslant 0$, so $u \in \omega(P)$. □

Corollary 8.1. *If the set* (8.5) *is bounded in* $L_\infty(\mathbb{R}_+; E)$, *the set* $\omega(P)$ *is contained in* $C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$.

Доказательство. Let $u \in \omega(P)$, then there exist sequences $t_m \to \infty$ and $\{u_m\} \subset P$ such that $\mathrm{T}(t_m)u_m \to u$ in $C(\mathbb{R}_+; E_0)$. The set P is bounded in $L_\infty(\mathbb{R}_+; E)$, so there exists a number $R > 0$ such that $\|v\|_{L_\infty(\mathbb{R}_+;E)} \leqslant R$ for any $v \in P$. The functions of class $C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$ are weakly continuous in E, so we have $\|v(t)\|_E \leqslant R$ for all $t \geqslant 0$ and $v \in P$. In particular it follows that the sequence $\{\mathrm{T}(t_m)u_m\}$ is bounded in $L_\infty(\mathbb{R}_+; E)$. Consequently, it converges to u $*$-weakly in $L_\infty(\mathbb{R}_+; E)$. Thus $u \in L_\infty(\mathbb{R}_+; E)$. This is true for any $u \in \omega(P)$, i. e. we have proved the inclusion

$$\omega(P) \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E).$$ □

□

Remark 8.1. If the set P is translation invariant, i. e. $\mathrm{T}(h)P \subset P$ $(h \geqslant 0)$, then for $s \geqslant t \geqslant 0$ we have $\mathrm{T}(s)P = \mathrm{T}(t)\mathrm{T}(s-t)P \subset \mathrm{T}(t)P$, so by formula (8.6) we get

$$\omega(P) = \bigcap_{t\geqslant 0} [\mathrm{T}(t)P].$$

Proposition 8.4. *Suppose that a trajectory space*

$$\mathcal{H}_\lambda^+ \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E), \tag{8.7}$$

is assigned to every λ *belonging to a metric space* Λ. *Suppose that each space* $\mathcal{H}_\lambda^+$ *has the minimal trajectory attractor of the form* $\mathcal{U}_\lambda = \omega(P_\lambda)$, *where the set* P_λ *is translation invariant (i. e.* $\mathrm{T}(h)P \subset P$ *for all* $h \geqslant 0$*) and besides* $P_\lambda \subset \mathcal{H}_\lambda^+ \cap P$ *where* P *is precompact in* $C(\mathbb{R}_+; E_0)$ *and bounded in* $L_\infty(\mathbb{R}_+; E)$. *Finally, suppose that the following condition holds:*

(C′) if $\lambda_m \to \lambda_0$, $v_m \in P_{\lambda_m}$, *and* $v_m \to v_0$ *in* $C(\mathbb{R}_+; E_0)$, *then* $v_0 \in [P_{\lambda_0}]$.

Then condition (C) of Proposition 8.1 holds.

Доказательство. Let $\lambda_m \to \lambda_0$, $u_m \in \mathcal{U}_{\lambda_m}$, $u_m \to u_0$ in $C(\mathbb{R}_+; E_0)$; we claim that $u_0 \in \mathcal{U}_{\lambda_0}$.

Since $u_m \in \mathcal{U}_{\lambda_m} = \omega(P_{\lambda_m})$, by Proposition 8.3 we see that there exist $w_m \in P_{\lambda_m}$ and $t_m \geqslant m$ such that

$$\|\mathrm{T}(t_m)w_m - u_m\|_{C(\mathbb{R}_+;E_0)} < \frac{1}{m}.$$

Then

$$\|\mathrm{T}(t_m)w_m - u_0\|_{C(\mathbb{R}_+;E_0)} \leqslant \|\mathrm{T}(t_m)w_m - u_m\|_{C(\mathbb{R}_+;E_0)} + \|u_m - u_0\|_{C(\mathbb{R}_+;E_0)} \to 0,$$

that is we have the following convergence in $C(\mathbb{R}_+; E_0)$:

$$\mathrm{T}(t_m)w_m \to u_0. \tag{8.8}$$

Let us show that $u_0 \in \omega(P_{\lambda_0})$. Take $t \geqslant 0$. Since $t_m \to \infty$, we eventually have $t_m - t \geqslant 0$. Put $v_m = \mathrm{T}(t_m - t)w_m$, then $v_m \in P_{\lambda_m}$ since $w_m \in P_{\lambda_m}$ and the set P_{λ_m} is translation invariant. The sequence $\{v_m\}$ lies in the set P, so it has a subsequence $\{v_{m_k}\}$ that converges in $C(\mathbb{R}_+; E_0)$ to a function v_0. According to condition (C$'$) we have $v_0 \in [P_{\lambda_0}]$. Translation operators are continuous in $C(\mathbb{R}_+; E_0)$, so

$$\mathrm{T}(t)v_{m_k} \to \mathrm{T}(t)v_0 \quad (k \to \infty).$$

On the other hand, it follows from (8.8) that

$$\mathrm{T}(t)v_{m_k} = \mathrm{T}(t)\mathrm{T}(t_{m_k} - t)w_{m_k} = \mathrm{T}(t_{m_k})w_k \to u_0.$$

Therefore we have $u_0 = \mathrm{T}(t)v_0 \in \mathrm{T}(t)\,[P_{\lambda_0}]$. Since t is arbitrary, we get

$$u_0 \in \bigcap_{t\geqslant 0} \mathrm{T}(t)\,[P_{\lambda_0}]\,.$$

By Remark 8.1, the right-hand side of the last inclusion coincides with the set $\omega(P_{\lambda_0}) = \mathcal{U}_{\lambda_0}$, i. e. $u_0 \in \mathcal{U}_{\lambda_0}$. This completes the proof. □

Corollary 8.2. *Under the hypothesis of Proposition 8.4 the limit relations* (8.2) *and* (8.4) *hold.*

The following assertion provides a sufficient condition for the minimal trajectory attractor of a trajectory space to be represented as an ω-limit set.

Proposition 8.5. *Suppose that a trajectory space*

$$\mathcal{H}^+ \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$$

has an absorbing set $P \subset \mathcal{H}^+$ that is precompact in $C(\mathbb{R}_+; E_0)$, bounded in $L_\infty(\mathbb{R}_+; E)$ and translation invariant, i. e. $\mathrm{T}(t)P \subset P$ for all $t \geqslant 0$. Then the set $\omega(P)$ is the minimal trajectory attractor of $\mathcal{H}^+$.

Доказательство. We claim that the set $\omega(P)$ is a semi-attractor of $\mathcal{H}^+$.

According to Remark 8.1 we have

$$\omega(P) = \bigcap_{t\geqslant 0}[\mathrm{T}(t)P]. \tag{8.9}$$

Since P is translation invariant and precompact, the right-hand side of (8.9) is the intersection of a centered family of nonempty closed subsets of the compact set $[P]$, whence $\omega(P)$ is a nonempty compact set in $C(\mathbb{R}_+; E_0)$.

Now let us prove that $\omega(P)$ is bounded in $L_\infty(\mathbb{R}_+;E)$. By (8.9) we have $\omega(P) \subset [P]$, so it suffices to show that $[P]$ is bounded. Let $u \in [P]$ and let a sequence $\{u_m\} \subset P$ converge in $C(\mathbb{R}_+;E_0)$ to u. Since P is bounded in $L_\infty(\mathbb{R}_+;E)$, there exists $R > 0$ such that $\|v\|_{L_\infty(\mathbb{R}_+;E)} \leqslant R$ for any $v \in P$. Consequently, the sequence $\{u_m\}$ is bounded in $L_\infty(\mathbb{R}_+;E)$, so it converges in the $*$-weak topology of $L_\infty(\mathbb{R}_+;E)$ as well. By a property of weak convergence we have

$$\|u\|_{L_\infty(\mathbb{R}_+;E)} \leqslant \varliminf_{m\to\infty} \|u_m\|_{L_\infty(\mathbb{R}_+;E)} \leqslant R.$$

This holds for any $u \in [P]$, so $[P]$ is bounded in $L_\infty(\mathbb{R}_+;E)$. Therefore $\omega(P)$ is bounded, too.

The translation invariance of $\omega(P)$ follows from the fact that the translation operators are continuous in $C(\mathbb{R}_+;E_0)$. Indeed, for any $h \geqslant 0$ we have

$$\mathrm{T}(h)\omega(P) = \mathrm{T}(h)\bigcap_{t\geqslant 0}[\mathrm{T}(t)P] \subset \bigcap_{t\geqslant 0}\mathrm{T}(h)[\mathrm{T}(t)P] \subset \bigcap_{t\geqslant 0}[\mathrm{T}(h)\mathrm{T}(t)P]$$
$$= \bigcap_{t\geqslant 0}[\mathrm{T}(t)\mathrm{T}(h)P] = \bigcap_{t\geqslant 0}[\mathrm{T}(t)P] = \omega(P).$$

Finally let us show that $\omega(P)$ is attracting. Assume the contrary. Then there exist $\varepsilon > 0$, a bounded set $B \subset \mathcal{H}^+$, a sequence $\{u_m\} \subset B$, and a sequence of numbers $t_m \to \infty$ such that

$$\operatorname{dist}(\mathrm{T}(t_m)u_m, \omega(P)) \geqslant \varepsilon. \tag{8.10}$$

Since P is absorbing and precompact, without loss of generality we can assume that $\{\mathrm{T}(t_m)u_m\} \subset P$ and $\mathrm{T}(t_m)u_m \to v$. It is readily seen that $v \in \omega(P)$ since for any $t \geqslant 0$ we eventually have $t_m > t$ and therefore

$$\mathrm{T}(t_m)u_m \in \bigcup_{s\geqslant t}\mathrm{T}(s)P,$$

and thus

$$v \in \left[\bigcup_{s\geqslant t}\mathrm{T}(s)P\right].$$

The latter inclusion holds for any $t \geqslant 0$, so $v \in \omega(P)$. On the other hand, passing to the limit in (8.10), we arrive at a contradiction:

$$\operatorname{dist}(v, \omega(P)) \geqslant \varepsilon.$$

This constradiction proves that $\omega(P)$ is attracting.

We have proved that $\omega(P)$ is a semi-attractor. By Theorem 7.1 the minimal trajectory attractor $\mathcal{U}$ of the trajectory space $\mathcal{H}^+$ exists.

It is known (see [9]) that the minimal trajectory attractor is the least trajectory semi-attractor with respect to inclusion, whence

$$\mathcal{U} \subset \omega(P). \tag{8.11}$$

Since $\mathcal{U}$ attracts P and the latter is translation invariant, we have

$$\sup_{v\in P}\inf_{u\in\mathcal{U}}\|v-u\|_{C(\mathbb{R}_+;E_0)} \leqslant \sup_{v\in \mathrm{T}(t)P}\inf_{u\in\mathcal{U}}\|v-u\|_{C(\mathbb{R}_+;E_0)}$$
$$\leqslant \sup_{w\in P}\inf_{u\in\mathcal{U}}\|\mathrm{T}(t)w-u\|_{C(\mathbb{R}_+;E_0)} \to 0 \quad (t\to\infty),$$

i. e. $P \subset [\mathcal{U}] = \mathcal{U}$ and consequently $[P] \subset \mathcal{U}$. However, $\omega(P) \subset [P]$, so

$$\omega(P) \subset \mathcal{U}.$$

Combining the last inclusion with (8.11) we see that $\omega(P)$ is the minimal trajectory attractor. □

Remark 8.2. Proposition 8.5 gives a hint how to construct trajectory spaces in such a way that their minimal trajectory attractors are ω-limit sets. Suppose we are given an evolutionary equation

$$v' = A(v),$$

and we have introduced the notion of its solution on $\mathbb{R}_+$. Following the usual approach, consider a set of solutions satisfying an estimate that makes it possible to construct an absorbing set $\widetilde{P}$ bounded in $L_\infty(\mathbb{R}_+; E)$ and precompact in $C(\mathbb{R}_+; E_0)$. Add to this 'preliminary' trajectory space all the solutions belonging to $\widetilde{P}$ and denote by $\mathcal{H}^+$ the enhanced trajectory space. Since the equation at issue is autonomous, we expect that the function $\mathrm{T}(h)v$ $(h \geqslant 0)$ is its solution if so is v. Hence $\mathcal{H}^+$ satisfies the hypothesis of Proposition 8.5 with $P = \mathcal{H}^+ \cap \widetilde{P}$.

8.1. Convergence of attractors of approximating problems in the polymer solution model

In this section we introduce trajectory spaces for the approximating problems in the model of polymer solutions and prove that their trajectory and global attractors converge to correspondent attractors of the unperturbed problem.

As before, we use the spaces $E = V^1$ $E_0 = V^{1-\delta}$ in order to introduce trajectory spaces.

Let us modify the definition of the trajectory space of problem (1.1)–(1.4) according to Remark 8.2. Let $\widetilde{R} \geqslant \max\{4R_0, 6R_1\}$, where R_0 R_1 are the constants involved in (3.3) and (6.1).

Definition 8.2. A function $v \in W_1^{\mathrm{loc}}(\mathbb{R}_+) \cap L_\infty(\mathbb{R}_+; E)$ is called a *trajectory* of problem (1.1)–(1.4) if it is a solution of the problem with certain $a \in V^1$ and satisfies either the estimate

$$\|v(t)\|_1 + \|v'(t)\|_{-1} \leqslant R_0 \left(1 + \|v\|^2_{L_\infty(\mathbb{R}_+, V^1)} e^{-\alpha t}\right) \quad \text{a. a. } t \geqslant 0 \tag{8.12}$$

or the estimate

$$\|v\|_{L_\infty(\mathbb{R}_+; V^1)} + \|v'\|_{L_\infty(\mathbb{R}_+; V^{-1})} \leqslant \widetilde{R} \quad \text{a. a. } t \geqslant 0. \tag{8.13}$$

The set of trajectories is called the *trajectory space* and denoted $\mathcal{H}_0^+$.

Remark 8.3. It is clear that the trajectory space $\mathcal{H}_0^+$ differs from the space $\mathcal{H}^+$ introduced by Definition 3.2 in the way that it contains all the solutions that satisfy (8.13). It follows that for any $a \in V^1$ there exists a trajectory $v \in \mathcal{H}_0^+$ such that $v(0) = a$.

Now we introduce trajectory spaces for equation (4.2). Note that it is not autonomous, since its left-hand side involves the coefficient e_α, which is independent of time. However, the theory of attractors of non-invariant spaces is versatile enough to be appied to non-autonomous equations.

Definition 8.3. The *trajectory space* $\mathcal{H}_\varepsilon^+$ of equation (4.2) is the set that consists of solutions of this equation that belong to $L_\infty(\mathbb{R}_+, \infty)$ and satisfy the estimate

$$\varepsilon \|v(0)\|_3^2 \leqslant 1, \tag{8.14}$$

as well as of functions $\mathrm{T}(h)v$, where v is a solution of equation (4.2) on $\mathbb{R}_+$, $h \geqslant 0$ and $\mathrm{T}(h)v$ satisfies the estimate

$$\|\mathrm{T}(h)v\|_{L_\infty(\mathbb{R}_+; V^1)} + \|\mathrm{T}(h)v'\|_{L_\infty(\mathbb{R}_+; V^{-1})} \leqslant \widetilde{R}. \tag{8.15}$$

Remark 8.4. Note that the construction of trajectories used in the proof of Theorem 3.2 involves solutions of the approximating problem whose initial value satisfies (8.14). This justifies inequality (8.14) in Definition 8.3.

The standard approach to attractors of nonautonomous equations involves families of trajectory spaces (see [9], [10]). Specifically, suppose we are given an evolutionary equation

$$v' = A_{\sigma(t)}(v) \qquad (\sigma \in \Sigma),$$

where the operator A depends on time via the intermediate function $\sigma(t)$ defined for a. a. $t \geqslant 0$ (such a function is called the *symbol* of the equation). Given a set Σ of possible σ's, a trajectory space $\mathcal{H}^+_\sigma \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$ is introduced for each σ, so that we have a family of trajectory spaces $\{\mathcal{H}^+_\sigma : \sigma \in \Sigma\}$. Consider the united space $\mathcal{H}^+_\Sigma = \bigcup_{\sigma\in\Sigma} \mathcal{H}^+_\sigma$. (Minimal) trajectory/global attractors of the latter space are called *uniform* attractors of the family $\{\mathcal{H}^+_\sigma : \sigma \in \Sigma\}$. In the particular case of one point set Σ this construction yields a single trajectory space and its attractors. For the sake of simplicity we formulate Definition 8.3 for an ordinary trajectory space.

Let the set $\widetilde{P}$ be defined by equation (7.1). Put $P_0 = \widetilde{P} \cap \mathcal{H}^+_0$.

Lemma 8.1. *The trajectory space $\mathcal{H}^+_0$ has the minimal trajectory attractor*

$$\mathcal{U}_0 = \omega(P_0). \tag{8.16}$$

Доказательство. Let us show that P_0 satisfies the hypothesis of Proposition 8.5.

The set P_0 is bounded in $L_\infty(\mathbb{R}_+; V^1)$ and precompact in $C(\mathbb{R}_+; V^{1-\delta})$, since it lies in the set $\widetilde{P}$, which has these properties (Lemma 7.1).

Now let us prove that P_0 is translation invariant. Take $h \geqslant 0$ and $v \in P_0$. The function v satisfies identity (3.1). This identity is autonomous, so the function $\mathrm{T}(h)v$ satisfies it as well. Besides,

$$\|\mathrm{T}(h)\|_{L_\infty(\mathbb{R}_+;V^1)} + \|\mathrm{T}(h)\|_{L_\infty(\mathbb{R}_+;V^{-1})} \leqslant \|v\|_{L_\infty(\mathbb{R}_+;V^1)} + \|v'\|_{L_\infty(\mathbb{R}_+;V^{-1})} \leqslant \widetilde{R},$$

and thus $\mathrm{T}(h)v \in P_0$.

Finally let us prove that P_0 is absorbing. Let the set $B \subset \mathcal{H}^+_0$ be bounded in $L_\infty(\mathbb{R}_+; V^1)$ and let R be so large that $\|v\|_{L_\infty(\mathbb{R}_+;V^1)}$ for any $v \in B$. Take t_0 such that $R^2 e^{-\alpha t_0} \leqslant 1$.

If $v \in B$ and $v \notin P_0$, then $v \notin \widetilde{P}$, and by definition of the trajectory space we see that v satisfies (8.14). It follows that for $t \geqslant t_0$ we have

$$\|v(t)\|_1 + \|v'(t)\|_{-1} \leqslant 2R_0,$$

whence for any $h \geqslant t_0$ we obtain

$$\|\mathrm{T}(h)v\|_{L_\infty(\mathbb{R}_+;V^1)} + \|\mathrm{T}(h)v'\|_{L_\infty(\mathbb{R}_+;V^{-1})} \leqslant 4R_0 \leqslant \widetilde{R}.$$

This means that $\mathrm{T}(h)v \in \widetilde{P}$, and since the function $\mathrm{T}(h)v$ satisfies identity (3.1), we have $\mathrm{T}(h)v \in \mathcal{H}^+_0$ and thus $\mathrm{T}(h)v \in P_0$.

On the other hand, if $v \in B$ and $v \in P_0$, we have $\mathrm{T}(h)v \in P_0$ for any $h \geqslant 0$ since P_0 is translation invariant.

This argument shows that for any $v \in B$ the inclusion $\mathrm{T}(t)v \in P_0$ holds at least for $t \geqslant t_0$. In other words, P_0 is absorbing.

To summarize, the set $\widetilde{P}$ satisfies the hypothesis of Proposition 8.5. By virtue of this proposition we see that $\mathcal{H}^+_0$ has the minimal trajectory attractor given by (8.16). □

Now consider the attractors of the trajectory space generated by the approximating equation. First of all notice that this trajectory space is well defined. Indeed, since the solutions of equation (4.2) belong to $C(\mathbb{R}_+; V^3)$ and we require that the trajectories should belong to $L_\infty(\mathbb{R}_+; V^1)$, the inclusion

$$\mathcal{H}^+_\varepsilon \subset C(\mathbb{R}_+; V^{1-\delta}) \cap L_\infty(\mathbb{R}_+; V^1)$$

holds. Further, the space $\mathcal{H}^+_\varepsilon$ is nonempty, since by Theorem 6.1 any function $a \in V^3$ such that $\varepsilon\|a\|_3^2 \leqslant 1$ is the beginning of a trajectory. Thus the trajectory space $\mathcal{H}^+_\varepsilon$ is well defined.

Put $P_\varepsilon = \mathcal{H}^+_\varepsilon \cap \widetilde{P}$.

Lemma 8.2. *The trajectory space $\mathcal{H}_\varepsilon^+$ $(\varepsilon > 0)$ has the minimal trajectory attractor*

$$\mathcal{U}_\varepsilon = \omega(P_\varepsilon). \tag{8.17}$$

Доказательство. Let us show that the set P_ε satisfies the hypothesis of Proposition 8.5. Clearly, P_ε is bounded in $L_\infty(\mathbb{R}_+; V^1)$ and precompact in $C(\mathbb{R}_+; V^{1-\delta})$, since P_ε lies in $\widetilde{P}$ and the latter set has these properties. Any trajectory belonging to P_ε is of the form $\mathrm{T}(h)v$, where v is a solution of equation (4.2). Since $\widetilde{P}$ is translation invariant, it contains the function $\mathrm{T}(s)\mathrm{T}(h)v$. Consequently this function belongs to $\mathcal{H}_\varepsilon^+ \cap \widetilde{P} = P_\varepsilon$. Thus P_ε is translation invariant. Let us prove that it is absorbing. Take a set $B \subset \mathcal{H}_\varepsilon^+$ bounded in $L_\infty(\mathbb{R}_+; V^1)$; to be definite, assume that $\|u\|_{L_\infty(\mathbb{R}_+;V^1)} \leqslant R$ for each $u \in B$. Let t_0 be such that $R^2 e^{-\alpha t_0} \leqslant 1$.

Consider an arbitrary trajectory $v \in B$. According to the definition of $\mathcal{H}_\varepsilon^+$ there are two possibilites.

1) Suppose that v is a solution of equation (4.2) and satisfies (8.14). By Theorem 6.1 it satisfies the inequality

$$\|v(t)\|_1 + \sqrt{\varepsilon} e^{-\alpha t/2} \|v(t)\|_3 + \|v'(t)\|_{-1} + \varepsilon e^{-\alpha t} \|v'(t)\|_3 \leqslant R_1 \left(1 + \left(\|v(0)\|_1^2 + \varepsilon \|v(0)\|_3^2\right) e^{-\alpha t}\right),$$

for a. a. $t > 0$. Combining it with (8.14), we obtain

$$\|v(t)\|_1 + \|v'(t)\|_{-1} \leqslant R_1 \left(2 + \|v(0)\|_1^2 e^{-\alpha t}\right).$$

It follows that for $t \geqslant t_0$ we have

$$\|v(t)\|_1 + \|v'(t)\|_{-1} \leqslant 3R_1,$$

so for any $s \geqslant t_0$ we get

$$\|\mathrm{T}(s)v\|_{L_\infty(\mathbb{R}_+;V^1)} + \|\mathrm{T}(s)v'\|_{L_\infty(\mathbb{R}_+;V^{-1})} \leqslant 6R_1 \leqslant \widetilde{R}.$$

This means that $\mathrm{T}(s)v \in \mathcal{H}_\varepsilon^+$ and since $\mathrm{T}(s)v \in \widetilde{P}$, we have the inclusion $\mathrm{T}(s)v \in P_\varepsilon$ for any $s \geqslant t_0$.

2) If v is a function of the form $\mathrm{T}(h)w$ and satisfies

$$\|v\|_{L_\infty(\mathbb{R}_+;V^1)} + \|v'\|_{L_\infty(\mathbb{R}_+;V^{-1})} \leqslant \widetilde{R},$$

then $v \in P_\varepsilon$ and therefore $\mathrm{T}(s)v \in P_\varepsilon$ for any $s \geqslant 0$, since P_ε is translation invariant.

Thus in any case $\mathrm{T}(s)v \in P_\varepsilon$ if $s \geqslant t_0$. This means that P_ε is absorbing.

By Proposition 8.5 the trajectory space $\mathcal{H}_\varepsilon^+$ has the minimal trajectory attractor, and formula (8.17) holds. □

Now we state the main result concerning the convergence of attractors.

Theorem 8.1. *Minimal trajectory attractors $\mathcal{U}_\varepsilon$ of equation* (4.2) *converge to the minimal trajectory attractor $\mathcal{U}_0$ of the trajectory space $\mathcal{H}_0^+$ of equation* (3.1) *in the sense of the Hausdorff semi-distance in $C(\mathbb{R}_+; V^{1-\delta})$, i. e.*

$$\sup_{u \in \mathcal{U}_\varepsilon} \inf_{v \in \mathcal{U}_0} \|u - v\|_{C(\mathbb{R}_+;V^{1-\delta})} \to 0 \qquad (\varepsilon \to 0).$$

Global attractors $\mathcal{A}_\varepsilon$ of approximating equation (4.2) *converge to the global attractor $\mathcal{A}_0$ of the trajectory space $\mathcal{H}_0^+$ of equation* (3.1) *in the sense of the Hausdorff semi-distance in the space $V^{1-\delta}$, i. e.*

$$\sup_{y \in \mathcal{A}_\varepsilon} \inf_{z \in \mathcal{A}_0} \|y - z\|_{V^{1-\delta}} \to 0 \qquad (\varepsilon \to 0).$$

Доказательство. Both assertions will follow from Corollary 8.2 if we prove that condition (C′) of Proposition 8.4 holds in our case. Specifically, we must prove that if $\varepsilon_m \to 0$, $v_m \in P_{\varepsilon_m}$, and $v_m \to v_0$ in $C(\mathbb{R}_+; V^{1-\delta})$, then v_0 belongs to the closure of P_0 with respect to the topology of $C(\mathbb{R}_+; V^{1-\delta})$.

Since $P_\varepsilon \subset \widetilde{P}$, the sequence $\{v_m\}$ lies in the set $\widetilde{P}$, which is compact in $C(\mathbb{R}_+; V^{1-\delta})$. It follows that $v_0 \in \widetilde{P}$.

The fact that v_0 is a solution of equation (3.1) is proved in much the same way as the passage to the limit in the proof of Theorem 3.2. Since $v_0 \in P$, then v_0 satisfies inequality (8.13) and thus $v_0 \in \mathcal{H}_0^+$. We have that $v_0 \in \mathcal{H}_0^+ \cap \widetilde{P} = P_0$, and (C′) holds. This concludes the proof. □

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