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OPERATOR IDEALS AND INTERPOLATION IN HILBERT COUPLES*

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Abstract: this paper is a survey of results on the operator ideals, which are related to interpolation theory of linear operators. We discuss the application of the real and complex method interpolation constructions to classical operator ideals, acting in scales of spaces, related to Hilbert couples, and some improvements of interpolation properties of linear operators, if these linear operators belong to some ideals. We consider also interpolation orbits with respect to some operator ideals. The paper is devoted to ideals of operators acting in Hilbert spaces or in couples of Hilbert spaces. We consider modern approach to the problem mentioned above.

Key words and phrases: cross-norm ideals, interpolation of linear operators.

ОПЕРАТОРНЫЕ ИДЕАЛЫ И ИНТЕРПОЛЯЦИЯ В ГИЛЬБЕРТОВЫХ ПАРАХ

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Аннотация: эта статья является обзором результатов об операторных идеалах, которые связаны с теорией интерполяции линейных операторов. Рассматривается применение интерполяционных конструкций вещественного и комплексного метода к классическим идеалам операторов, действующих в шкалах пространств, связанных с гильбертовыми парами, и уточнение интерполяционных свойств операторов, когда они принадлежат операторным идеалам. Мы также рассматриваем интерполяционные орбиты относительно некоторых операторных идеалов. Рассматриваются только операторные идеалы, действующие в гильбертовых пространствах или парах гильбертовых пространств. В обзоре рассмотрены современные подходы к решению упомянутых выше задач.

Ключевые слова: операторные идеалы, интерполяция линейных операторов.

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1. INTRODUCTION

Interpolation spaces and operator ideals are related to each other in at least two directions. First, you are able to apply an interpolation construction to classical operator ideals and to obtain a new operator ideal. If you obtain somewhat familiar ideal as a result, then you find a dependence between classical ideals, which can be useful. And this phenomenon was historically the first, which was performed in the works of I.C.Gohberg and M.G.Krein (see [8]). The second direction is to use somewhat complementary properties of operators from an ideal in order to get more sharp interpolation properties of these operators.

Thus the first direction is one of applications of the interpolation theory. The second direction can be considered as a part of interpolation theory itself. We are going to deal with both these relations between interpolation spaces and operator ideals in the given survey paper.

The most developed part of the operator ideals theory is the theory of cross-norm ideals of operators in Hilbert spaces. And for these ideals we have the most complete results concerning interpolation properties of operators from these ideals.

No proofs are discussed. The corresponding references are presented as well as several open problems are mentioned.

2. CLASSICAL IDEALS OF OPERATORS MAPPING HILBERT SPACES

We shall suppose that readers are familiar with main fundamentals of the theory of cross-norm ideals of operators mapping Hilbert spaces. In this Section we simply recall more or less common notations.

Let T be a bounded linear operator mapping a Hilbert space H into a Hilbert space F. This will be denoted by $T : H \to F$ and $T \in L(H \to F)$, where $L(H \to F)$ is the space of all bounded linear operators mapping H into F.

Singular numbers or s-numbers of $T \in L(H \to F)$ are defined as follows

$$s_n(T) = \inf_K \|T - K\|_{L(H\to F)},$$

where *infimum* is taken over all $K \in L(H \to F)$ provided dim $K(H) < n$. Thus $s_1(T) = \|T\|_{L(H\to F)}$ and $s_n(T) \geqslant s_{n+1}(T)$ for any $n \in \mathbb{N}$.

If the operator T is compact (which is denoted by $T \in S_\infty(H \to F)$), then $|T| = (T^*T)^{1/2}$ is also compact, therefore it's spectrum is discrete. Hence we are able to construct a monotone positive sequence $\lambda_j(|T|)$, which runs over all eigenvalues of $|T|$, taking into account it's multiplicities, and tends to 0. It turns out that $s_j(T) = \lambda_j(|T|)$ for any $j \geqslant 1$.

Thus $s_j(T) \to 0$, if $T \in S_\infty(H \to F)$. Moreover the subspace $S_\infty(H \to F) \subset L(H \to F)$ can be characterized by this property of s–numbers, i.e., $T \in S_\infty(H \to F)$ is equivalent to $s_j(T) \to 0$ or $\{s_j(T)\} \in c_0$.

The properties of s-numbers are well known (see [8]).

The s-numbers of compact operators are also used in *Schmidt's expansion* of these operators. Namely, for any $T \in S_\infty(H \to F)$ there exists an orthonormal sequence $\{\psi_j\}_{j=1}^d \quad (\psi_j \in F)$ and an orthonormal sequence $\{\varphi_j\}_{j=1}^d \quad (\varphi_j \in H)$, where $d = \min(\dim H, \dim F)$, such that

$$Tx = \sum_{j=1}^{d} s_j(T)(x, \varphi_j)\psi_j. \tag{2.1}$$

The Neumann–Schatten classes $S_p(H \to F)$, where $0 < p < \infty$, are defined as follows.

An operator T is said to be an element of $S_p(H \to F)$ if $\{s_j(T)\} \in l_p$ or

$$\sum_{j=1}^{\infty} s_j(T)^p < \infty.$$

It is known that $S_p(H \to F)$ is a Banach space, and

$$\|T\|_{S_p(H\to F)} = \left(\sum_{j=1}^{\infty} s_j(T)^p\right)^{1/p}.$$

is a norm on $S_p(H \to F)$, if $1 \leqslant p \leqslant \infty$. If $0 < p < 1$, then $\|T\|_{S_p(H\to F)}$ is a quasi-norm and $S_p(H \to F)$ is a quasi-Banach space.

The space $S_p(H \to F)$ is an ideal in $L(H \to F)$, which means that for any $U_1 \in L(H' \to H)$, $U_2 \in L(F \to F')$ and $T \in S_p(H \to F)$ we have $U_2TU_1 \in S_p(H' \to F')$ and

$$\|U_2TU_1\|_{S_p(H'\to F')} \leqslant \|U_2\| \cdot \|T\|_{S_p(H\to F)} \cdot \|U_1\|.$$

The ideal $S_1(H \to F)$ is one of the most important because it coincides with the ideal of all nuclear operators mapping H into F. The Neumann-Schatten classes are particular example of cross-norm ideals mapping Hilbert spaces.

Recall that an ideal $I(H)$ in the algebra $L(H)$ is called a cross-norm ideal or a symmetrically normed ideal if $I(H)$ is a Banach space with respect to a norm on $I(H)$ such that

$$\|UTS\|_{I(H)} \leqslant \|U\|_{L(H)}\|T\|_{I(H)}\|S\|_{L(H)}$$

for any U, $S \in L(H)$ and $T \in I(H)$.

Note that the notion of a cross-norm ideal can be easily extended to operators, mapping one Hilbert space to another.

3. BANACH COUPLES AND FUNCTORS

In this Section we recall simple general properties of Banach couples, which will be applied later on to couples of operator spaces. For detailed information see [4], [17].

Recall that if $\{X_0, X_1\}$ is a Banach couple, then for any $x \in X_0 + X_1$ and any s, $t > 0$ we can define the K-functional:

$$K(s, t, x; \{X_0, X_1\}) = \inf_{x=x_0+x_1} s\|x_0\|_{X_0} + t\|x_1\|_{X_1}$$

and $K(t, x; \{X_0, X_1\}) = K(1, t, x; \{X_0, X_1\})$.

3.1 Interpolation functors on couples of operator spaces

Definition 3.1. *Let $\overline{E} = \{E_0, E_1\}$ and $\overline{F} = \{F_0, F_1\}$ be Banach couples and $a \in E_0 + E_1$, $b \in F_0 + F_1$. The elements a and b are called orbitally equivalent with respect to the couples $\overline{E}$ and $\overline{F}$ if there exist linear operators $T : \overline{E} \to \overline{F}$ and $S : \overline{F} \to \overline{E}$ such that $Ta = b$, $Sb = a$.*

If any element $a \in E_0 + E_1$ is orbitally equivalent to some element $b \in F_0 + F_1$ then the couple $\overline{E}$ is called a partial retract of the couple $\overline{F}$.

If the couple $\overline{E}$ is a partial retract of the couple $\overline{F}$ and the couple $\overline{F}$ is a partial retract of the couple $\overline{E}$, then couples $\overline{E}$ and $\overline{F}$ are called orbitally equivalent.

The notion of orbital equivalence of Banach couples is rather similar to the notion of the homotopic equivalence of topological spaces, especially if we compare with the notion of isomorphism. For instance any couples of spaces $\{E_0, E_1\}$, where $E_0 = E_1$ with equivalent norms, are orbitally equivalent. Moreover $\{E_0, E_1\}$ is orbitally equivalent to the couple of one-dimensional spaces $\{\mathbb{R}, \mathbb{R}\}$.

Interpolation for couples of operator ideals are based mainly on the following theorem (see [32], [34]).

Theorem 3.1. *Banach couples $\{S_1(H \to F), L(H \to F)\}$ and $\{l_1, l_\infty\}$ are orbitally equivalent if both H and F are infinite dimensional. In general case any operator $T \in L(H \to F)$ is orbitally equivalent to the sequence $\{s_j(T)\} \in l_\infty$ with respect to the couples*

$$\{S_1(H \to F),\, L(H \to F)\} \qquad \text{and} \qquad \{l_1,\, l_\infty\}.$$

Moreover the operators mapping T into $\{s_j(T)\}$ and vice versa can be chosen with the unit norms in the corresponding spaces.

The proof is based on Schmidt's expansion (2.1) if we note that the subspace of operators $\{T_\lambda\} \subset L(H \to F)$, where $\lambda \in l_\infty$ such that

$$T_\lambda(x) = \sum_{j=1}^{\infty} \lambda_j (x,\, \varphi_j)\psi_j,$$

and the orthonormal sequences $\{\varphi_j\} \subset H$ and $\{\psi_j\} \subset F$ are fixed, is complemented in $L(H \to F)$ and corresponding projection is bounded in $S_1(H \to F)$.

Theorem 3.1 enables us to establish the one to one correspondence between interpolation spaces of the couple $\{S_1(H \to F),\, L(H \to F)\}$ and interpolation spaces of the couple $\{l_1,\, l_\infty\}$ if both H and F are infinite dimensional spaces.

Indeed if Φ is an interpolation space between l_1 and l_∞, consider the space $\widetilde{\Phi}$ between $S_1(H \to F)$ and $L(H \to F)$ consisting of all $T \in L(H \to F)$ which are orbitally equivalent to elements of Φ with respect to the couples $\{l_1,\, l_\infty\}$ and $\{S_1(H \to F),\, L(H \to F)\}$.

The norm on $\widetilde{\Phi}$ can be introduced by $\|T\|_{\widetilde{\Phi}} = \|\{s_j(T)\}\|_{\Phi}$.

It is easily seen that the correspondence between Φ and $\widetilde{\Phi}$ is an order isomorphism between sets of all interpolation spaces of the corresponding couples.

This one to one correspondence allows us to carry on interpolation properties of sequence spaces to the properties of corresponding operator spaces.

For instance, if two interpolation functors coincide on the couple $\{l_1,\, l_\infty\}$ then they coincide on the couple $\{S_1(H \to F),\, L(H \to F)\}$ and vice versa. Moreover if Φ_0 and Φ_1 are two interpolation spaces between l_1 and l_∞, then for any interpolation functor $\mathcal{F}$

$$\mathcal{F}(\widetilde{\Phi}_0,\, \widetilde{\Phi}_1) \,=\, \widetilde{\mathcal{F}(\Phi_0,\, \Phi_1)}. \tag{3.1}$$

For quasi-Banach ideals S_p we also have somewhat analogous results, but only for real method functors (mainly for the Lions-Peetre construction) (see [33]). This fact is likely related to the orbital equivalence of couples $\{l_p,\, l_\infty\}$ and $\{S_p(H),\, L(H)\}$ for any $0 < p < 1$ with respect to quasi-linear operators.

Definition 3.2. *A cross-norm ideal Φ of operators mapping Hilbert spaces is called an interpolation ideal if Φ is an interpolation space between $S_1(H \to F)$ and $L(H \to F)$.*

Any separable cross-norm ideal is an interpolation ideal as well as any ideal dual to some separable ideal (see [9]). In particular any the Neumann–Schatten ideal is an interpolation ideal.

Recall that any interpolation ideal Φ corresponds to an interpolation space between l_1 and l_∞ which is orbitally equivalent to Φ.

3.2 Couples of Hilbert spaces

Let $\overline{H} = \{H_0, H_1\}$ be a Hilbert couple, i.e., Banach couple, where H_0 and H_1 are Hilbert spaces. It is worthwhile to note that the intersection $H_0 \cap H_1$ and the sum $H_0 + H_1$ are also Hilbert spaces. For example we can introduce a quadratic form

$$\|x\|^2_{H_0 \cap H_1} = \|x\|^2_{H_0} + \|x\|^2_{H_1},$$

on the $H_0 \cap H_1$, and a quadratic form

$$\|x\|^2_{H_0 + H_1} = \inf_{x = x_0 + x_1} \|x_0\|^2_{H_0} + \|x_1\|^2_{H_1}$$

on the sum of spaces H_0 and H_1.

As we shall see below the isomorphic structure of an arbitrary Hilbert couple is not so complicated. It doesn't look very unexpected because of a simple structure of an arbitrary Hilbert space.

A Banach couple $\overline{X} = \{X_0, X_1\}$ is called regular if the space $X_0 \cap X_1$ is dense in X_0 and X_1.

In what follows we mainly consider regular Hilbert couples.

Any Hilbert couple generates an indefinite metric on $H_0 \cap H_1$

$$\Delta(x) = \|x\|^2_{H_0} - \|x\|^2_{H_1},$$

which is bounded by $\|x\|^2_{H_0 \cap H_1}$. Evidently $|\Delta(x)| < \|x\|^2_{H_0 \cap H_1}$, if $x \neq 0$.

It is easy to see also that any continuous indefinite metric $\mathcal{J}(x)$ on a Hilbert space H such that $|\mathcal{J}(x)| < \|x\|^2$ if $x \neq 0$ generates a Hilbert couple by introducing two new norms on the underlying Hilbert space, namely

$$\|x\|^2_{H_0} = \|x\|^2 - \mathcal{J}(x)$$
$$\|x\|^2_{H_1} = \|x\|^2 + \mathcal{J}(x).$$

The spaces H_0 and H_1 are now defined by completion with respect to the norms introduced. Thus we obtain a regular Hilbert couple such that $H_0 \cap H_1 = H$.

Let us denote by D a self-adjoined operator, generating the metric $\Delta(x)$, i.e.,

$$\Delta(x) = (Dx, x),$$

where (x, x) is the metric on $H_0 \cap H_1$. Hence the metric on H_0 is defined by

$$\frac{1}{2}((I + D)x, x) = \|x\|^2_{H_0},$$

and the metric on H_1 is defined by

$$\frac{1}{2}((I - D)x, x) = \|x\|^2_{H_1}.$$

Therefore, if the metric $\|x\|^2_0$ is fixed, then $\|x\|^2_1$ is introduced by the operator

$$\left(\frac{I - D}{I + D}\right)^{1/2} = A, \text{ that is } \|x\|^2_1 = (A^2 x, x)_0.$$

If E_λ denotes the spectral resolution of the operator A, then denote by G_n the space $(E_{2^{-n}} - E_{2^{-n-1}})(H_0)$ for any $n \in \mathbb{Z}$. Thus $H_0 = l_2(G_n)$, where $l_2(G_n)$ denotes the space of sequences $\{\xi_n\}_{-\infty}^{\infty}$ such that $\xi_n \in G_n$ and

$$\sum_{n=-\infty}^{\infty} \|\xi_n\|^2_{G_n} < \infty.$$

(Note that some G_n may be trivial, i.e., $G_n = 0$.)

The space H_1 now can be identified (up to equivalent norms) with the space $l_2(2^{-n}G_n)$ of the sequences $\{\xi_n\}$, such that

$$\sum_{n=-\infty}^{\infty} (2^{-n}\|\xi_n\|_{G_n})^2 < \infty.$$

Hence we are able to consider any regular Hilbert couple $\{H_0, H_1\}$ as a couple of vector-valued weight sequence spaces $\{l_2(G_n), l_2(2^{-n}G_n)\}$.

Such couples are well studied (see [3] Chapter 4), and the classical interpolation functors are described for these couples.

The Lions-Peetre construction $\overline{X}_{\theta,p}$, where $0 < \theta < 1$ and $0 < p \leqslant \infty$, applied to the couple $\{l_2(G_n), l_2(2^{-n}G_n)\}$ gives us a weighted l_p space, namely (see [3])

$$(l_2(G_n),\, l_2(2^{-n}G_n))_{\theta,p} = l_p(2^{-\theta n}G_n),$$

where $\{\xi_n\} \in l_p(2^{-\theta n}G_n)$ means $\{2^{-\theta n}\|\xi_n\|_{G_n}\} \in l_p$.

The Calderon complex method $[\overline{X}]_\theta$, where $0 < \theta < 1$ gives us the unique scale of Hilbert spaces connecting $l_2(G_n)$ and $l_2(2^{-n}G_n)$, i.e.,

$$[l_2(G_n),\, l_2(2^{-n}G_n)]_\theta = l_2(2^{-\theta n}G_n).$$

Moreover any interpolation Hilbert space H between $l_2(G_n)$ and $l_2(2^{-n}G_n)$ can be identified (see [17]) with the space $l_2(\varphi(1, 2^{-n})G_n)$, where $\varphi(s,t)$ is an interpolation function.

Recall that a function $\varphi(s,t) > 0$, $(s,\, t > 0)$, is called an interpolation function if

1) $\varphi(s,\, t)$ increases in s and t,

2) $\varphi(\lambda s,\, \lambda t) = \lambda\varphi(s,t)$ for any $\lambda, s, t > 0$.

The function $\varphi(s,t)$ is an interpolation function, if and only if $\varphi(1,t)$ is increasing and $\varphi(1,t)/t$ is decreasing, i.e., $\varphi(1,t)$ is a quasi-concave function on $(0, \infty)$.

If the couple of Hilbert spaces is presented in the form of a couple of vector-valued sequence spaces $\{l_2(G_n), l_2(2^{-n}G_n)\}$ then operators, mapping intermediate spaces between H_0 and H_1 can be presented in a matrix form. Indeed, if $T : H_0 \cap H_1 \to H_0 + H_1$, then there corresponds a block-matrix $\{T_{ij}\}$ such that $T_{ij} : G_j \to G_i$ and

$$T_{ij}(\xi) = (T(\xi e_j))_i,$$

where $e_j = \{\delta_{ij}\}_{i=-\infty}^{\infty}$, and $\xi \in G_j$.

Any weight space

$$l_2(w_nG_n) = \{\xi_n;\ \{\xi_n w_n\} \in l_2(G_n)\}$$

is isomorphic to $l_2(G_n)$, where the isometry is established with the help of the multiplication $M_w : \{\xi_n\} \mapsto \{w_n\xi_n\} \in l_2(G_n)$. This gives us an opportunity to present any condition

$$T \in S_p(l_2(w_nG_n) \to l_2(w_nG_n)), \tag{3.2}$$

as

$$M_w T M_{w^{-1}} \in S_p(l_2(G_n) \to l_2(G_n)).$$

If we use the matrix representation of the corresponding operators, then any condition (3.2) we reduce to a weight condition

$$\{\frac{w_i}{w_j}T_{ij}\} \in S_p(l_2(G_n) \to l_2(G_n)).$$

Thus we able to consider $S_p(l_2(w_nG_n) \to l_2(w_nG_n))$ as a weight space

$$S_p(l_2(G_n) \to l_2(G_n))(w_i w_j^{-1}). \tag{3.3}$$

3.3 Duality for Neumann-Schatten ideals mapping Hilbert couples

Consider the space $\mathfrak{m}$ of matrices $\{T_{ij}\}$, where T_{ij} are finite-rank operators mapping G_j into G_i and only finite number of $T_{ij} \neq 0$, i.e., $\mathrm{supp}_{i,j}\, T_{ij}$ is finite. There is a duality between this space and the space $\mathfrak{M}$ of arbitrary matrices $T_{ij} \in L(G_j \to G_i)$ which is established by the scalar product

$$\langle S, T\rangle = \sum_{i,j\in\mathbb{Z}} tr(S_{ij}T_{ji}),$$

where $S \in \mathfrak{m}$, $T \in \mathfrak{M}$.

This duality allows us to obtain an isometry between the conjugate space $S_p(l_2(G_n) \to l_2(G_n))^*$ for any $1 \leqslant p \leqslant \infty$ and the space $S_q(l_2(G_n) \to l_2(G_n))$, where $1/p + 1/q = 1 \quad (p \neq 1)$ (see [8]) , and

$$S_1(l_2(G_n) \to l_2(G_n))^* \cong L(l_2(G_n) \to l_2(G_n)).$$

The same duality applied to the weight space (3.3) gives us the isometry

$$S_p(l_2(w_nG_n) \to l_2(w_nG_n))^* = S_p(l_2(G_n) \to l_2(G_n))(w_iw_j^{-1})^*$$

$$\cong S_q(l_2(G_n) \to l_2(G_n))(w_jw_i^{-1}) = S_q(l_2(w_n^{-1}G_n) \to l_2(w_n^{-1}G_n)).$$

3.4 Spectral classification of Hilbert couples

A Banach couple $\overline{X} = \{X_0,\, X_1\}$ is called K-abundant (see [4]), if for any interpolation function $\varphi(s,t)$ there exists an element $a \in X_0 + X_1$, such that

$$\varphi(s,t) \asymp K(s,t,a;\, \{X_0,\, X_1\}).$$

(Recall that $\asymp$ means as usual that there exist two positive constants c and d such that

$$c\,\varphi(s,t) \leqslant K(s,t,a;\, \{X_0,\, X_1\}) \leqslant d\,\varphi(s,t)$$

for all $s, t > 0$.

The couple $\overline{X}$ is called K_0-abundant if the same is true for any function φ such that $\varphi(1,t) \to 0$ as $t \to 0$ and $\varphi(s,1) \to 0$ as $s \to 0$.

N. Krugliak found (see, [4]) that $\overline{X}$ is K_0-abundant if for some $0 < \theta < 1$

$$K(t, a_\theta; \{X_0,\, X_1\}) \asymp t^\theta,$$

where $a_\theta \in X_0 + X_1$. Thus this property is simple in checking. In particular if $G_n \neq 0$ for any $n \in \mathbb{Z}$, then for any $a_\theta = \{2^{n\theta}g_n\}_{n=-\infty}^{\infty}$ with $\|g_n\|_{G_n} = 1$ we easily have

$$K(t, a_\theta;\, \{l_2(G_n),\, l_2(2^{-n}G_n)\}) \asymp \left(\sum_{n=-\infty}^{\infty} (\min(1,\, t2^{-n})2^{n\theta}\|g_n\|_{G_n})^2\right)^{1/2}$$

$$= \left(\sum_{n=-\infty}^{\infty} (\min(1,\, t2^{-n})2^{n\theta})^2\right)^{1/2} \asymp t^\theta.$$

Therefore any couple, where $G_n \neq 0$, is K_0-abundant.

This remark gives us some right to call K_0-abundant Hilbert couples as spectrally filling couples or couples with full spectrum. Any ordered couple, i.e., the couple for which $H_0 \subset H_1$ or $H_0 \supset H_1$ is not spectrally filling. However any ordered couple $\overline{H}$ may happen to be a "half" of a spectrally filling couple, i.e., the couple $\overline{H} \oplus \overline{F}$ is spectrally filling for some ordered couple $\overline{F}$. In this case we call the couple $\overline{H}$ as a generalized spectrally filling couple.

Unfortunately there is no an adequate notion of spectrum of a Banach or Hilbert couple. The problem is that the exact norms of spaces involved are not very important. Recall that even the famous Riesz–Thorin interpolation theorem is not exact in general in spaces of real valued functions. Hence the notion of spectrum of Banach couple has to be independent of norms involved up to equivalence of norms. For example, it may happen that orbitally equivalent couples have identical spectrum.

Note that a lot of Hilbert couples in Analysis are generalized spectrally filling.

4. INTERPOLATION THEOREMS FOR IDEALS MAPPING HILBERT COUPLES

Interpolation theorems for operators belonging to the Neumann–Schatten classes can be sharper then interpolation results for bounded operators.

Let us consider the Lions–Peetre interpolation functor $(X_0, X_1)_{\theta,p}$. Recall that if $\{X_0, X_1\}$ is a Banach couple, $0 < \theta < 1$, $0 < p \leqslant \infty$, then $(X_0, X_1)_{\theta,p}$ by definition consists of $x \in X_0 + X_1$ such that $\{2^{-n\theta} K(2^n, x, \{X_0, X_1\})\} \in l_p$, equipped with the natural quasi-norm.

Let us consider Hilbert couples $\{H_0, H_1\}$ and $\{F_0, F_1\}$. Then standard interpolation gives us that $T : \{H_0, H_1\} \to \{F_0, F_1\}$ implies $(H_0, H_1)_{\theta,r} \to (F_0, F_1)_{\theta,r}$. As we see below this result is improved if $T \in S_{p_0}(H_0 \to F_0) \cap S_{p_1}(H_1 \to F_1)$.

Theorem 4.1. *[16] If $T \in S_{p_0}(H_0 \to F_0)$ and $T \in S_{p_1}(H_1 \to F_1)$, where $0 < p_0, p_1 \leqslant \infty$, then for any $0 < r \leqslant \infty$ we have*

$$T : (H_0,\, H_1)_{\theta,r} \to (F_0,\, F_1)_{\theta,q},$$

where $0 < \theta < 1$ and $1/q = 1/r + (1-\theta)/p_0 + \theta/p_1$.

(Note that we obtain the same result if replace $S_\infty(H_i \to F_i)$ for $i = 0,\,1$ by $L(H_i \to F_i)$.)

The natural multiplicative inequality also takes place, i.e.,

$$\|T\|_{(H_0,\,H_1)_{\theta,r} \to (F_0,\,F_1)_{\theta,q}} \leqslant C\, \|T\|^{1-\theta}_{S_{p_0}(H_0 \to F_0)} \cdot \|T\|^{\theta}_{S_{p_1}(H_1 \to F_1)},$$

where C depends on p_0, p_1, r, θ.

This interpolation theorem is optimal from the orbital point of view. Namely if the couple $\{H_0,\, H_1\}$ is spectrally filling, then for any $y \in (F_0,\, F_1)_{\theta,q}$ there exists $x \in (H_0,\, H_1)_{\theta,r}$ and an operator $T \in S_{p_0}(H_0 \to F_0) \cap S_{p_1}(H_1 \to F_1)$ such that $y = Tx$.

This Theorem 4.1 is intimately connected with the following "model" interpolation theorem for discrete l_p spaces. In [18] Theorem 4.1 was used in the proof of this "model" theorem.

Theorem 4.2. *If a linear operator $T : \{l_{s_0},\, l_{s_1}(2^{-n})\} \to \{l_{t_0},\, l_{t_1}(2^{-n})\}$, where $0 < s_0,\, s_1,\, t_0,\, t_1 \leqslant \infty$ then*

1) $T : l_r(2^{-n\theta}) \to l_q(2^{-n\theta})$, where $1/q = 1/r + (1-\theta)(1/t_0 - 1/s_0)_+ + \theta(1/t_1 - 1/s_1)_+$ and $0 < r \leqslant \infty,\ 0 < \theta < 1$;

2) $T \in S_p(l_2(2^{-n\theta}) \to l_2(2^{-n\theta}))$, where $1/p = (1-\theta)(1/t_0 - 1/s_0)_+ + \theta(1/t_1 - 1/s_1)_+$, and S_∞ has to be replaced by L.

The proof of both these results is based on the Grothendieck factorization theorem (see [13]) and the following theorem on coherent factorization in Hilbert couples (see [16]).

Two operators are said to be metrically equivalent if they have identical s-numbers.

Theorem 4.3. *Let an operator* $T : \{H_0, H_1\} \to \{F_0, F_1\}$ *and* T *be factorized through Hilbert spaces* $\widetilde{F}_0$ *and* $\widetilde{F}_1$ *separately as an operator mapping* H_0 *into* F_0 *and as an operator mapping* H_1 *into* F_1, *i.e.,* $T|_{H_0} = U_0S_0$, *where* $S_0; H_0 \to \widetilde{F}_0$, $U_0 : \widetilde{F}_0 \to F_0$, *and* $T|_{H_1} = U_1S_1$, *where* $S_1; H_1 \to \widetilde{F}_1$, $U_1 : \widetilde{F}_1 \to F_1$.

Then there exist a Hilbert couple $\{Q_0, Q_1\}$ *and operators* $S : \{H_0, H_1\} \to \{Q_0, Q_1\}$ *and* $U : \{Q_0, Q_1\} \to \{F_0, F_1\}$ *such that* $T = US$ *and* $U|_{Q_i}$ *is metrically equivalent to* U_i *and* $S|_{H_i}$ *is metrically equivalent to* S_i *for* $i = 0, 1$.

Let us illustrate this theorem by the remark on opportunity to separate the conditions $T \in S_{p_0}(H_0 \to F_0)$ and $T \in S_{p_1}(H_1 \to F_1)$. Indeed $T|_{H_0} = I \cdot T$ and $T|_{H_1} = T \cdot I$, where $\widetilde{F}_0 = F_0$ and $\widetilde{F}_1 = H_1$. Therefore, $T = US$, where $U : \{Q_0, Q_1\} \to \{F_0, F_1\}$ and $S : \{H_0, H_1\} \to \{Q_0, Q_1\}$, and $U \in S_{p_1}(Q_1 \to F_1) \cap L(Q_0 \to F_0)$ and $S \in S_{p_0}(H_0 \to Q_0) \cap L(H_1 \to Q_1)$.

5. ORBITS WITH RESPECT TO IDEALS IN HILBERT COUPLES

The notion of a norm ideal of operators mapping Hilbert couples is not uniquely defined (see [12], [7]). Here we use only the following one (see [7], [17]), which naturally corresponds to cross-norm ideals.

Definition 5.1. *The Banach space* $I(\overline{H} \to \overline{F}) \subset L(\overline{H} \to \overline{F})$ *is called an ideal mapping Hilbert couples if*

1) $\|T\|_{L(\overline{H}\to\overline{F})} \leqslant \|T\|_{I(\overline{H}\to\overline{F})}$,

2) *any rank-one operator* $T \in L(\overline{H} \to \overline{F})$ *belongs to* $I(\overline{H} \to \overline{F})$ *provided*

$$\|T\|_{L(\overline{H}\to\overline{F})} = \|T\|_{I(\overline{H}\to\overline{F})},$$

3) *for any* $T \in I(\overline{H} \to \overline{F})$ *and any* $S \in L(\overline{H} \to \overline{H})$, $U \in L(\overline{F} \to \overline{F})$ *we have* $UTS \in I(\overline{H} \to \overline{F})$ *provided* $\|UTS\|_{I(\overline{H}\to\overline{F})} \leqslant \|U\|_{L(\overline{F}\to\overline{F})}\|T\|_{I(\overline{H}\to\overline{F})}\|S\|_{L(\overline{H}\to\overline{H})}$.

Recall also the notion of interpolation orbit of an element $a \in H_0 + H_1$ with respect to some ideal of operators mapping Hilbert couples.

Definition 5.2. *Let* $I(\overline{H} \to \overline{F})$ *be an ideal, and* $a \in H_0 + H_1$. *The space of all* $y \in F_0 + F_1$ *such that* $y = Ta$, *where* $T \in I(\overline{H} \to \overline{F})$, *is called an orbit of* a *with respect to* $I(\overline{H} \to \overline{F})$ *and is denoted by* $Orb\,(a;\, I(\overline{H} \to \overline{F}))$.

The orbit is equipped with the norm

$$\|y\| = \inf_{\substack{T \\ y = Ta}} \|T\|_{I(\overline{H}\to\overline{F})}\|a\|_{H_0+H_1}.$$

If we use this definition of an ideal, then the space $Orb\,(a;\, I(\overline{H} \to \overline{F}))$ turns out be an intermediate space between F_0 and F_1. Moreover it is an interpolation space between F_0 and F_1.

It is easily seen that each space of the form $S_{p_0}(H_0 \to F_0) \cap S_{p_1}(H_1 \to F_1)$ is an ideal in $L(\overline{H} \to \overline{F})$.

We are going to describe interpolation orbits of arbitrary $a \in H_0 + H_1$ with respect to $S_{p_0}(H_0 \to F_0) \cap S_{p_1}(H_1 \to F_1)$ for any $0 < p_0, p_1 \leqslant \infty$.

5.1 The generalized Lions–Peetre spaces of means

Let $\varphi(s,t)$ be interpolation function. In what follows we also suppose that $\varphi(s,1) \to 0$ and $\varphi(1,t) \to 0$ as $s \to 0$ and $t \to 0$. As usual this property is denoted as $\varphi \in \Phi_0$.

Recall that for any function $\varphi \in \Phi_0$ there exists so called balanced sequence $\{u_m\}_{m\in\mathbb{M}}$, where $\mathbb{M}$ is an interval of integers [5]. The balanced sequence is not uniquely defined. There are several ways to construct a balanced sequence for a given function $\varphi \in \Phi_0$, (see [11], [4], [5]). For instance it may be constructed inductively by $u_0 = 1$ and

$$\min\left(\frac{\varphi(1,u_{m+1})}{\varphi(1,u_m)}, \frac{u_{m+1}\varphi(1,u_m)}{u_m\varphi(1,u_{m+1})}\right) = 2,$$

where $\mathbb{M}$ is the maximal interval in $\mathbb{Z}$ such that m and $m+1 \in \mathbb{M}$.

Definition 5.3. *[22] Let $\{X_0,\, X_1\}$ be a couple of Banach spaces, $\varphi \in \Phi_0$, and $1 \leqslant p_0, p_1 \leqslant \infty$. The space $\varphi(X_0,\, X_1)_{p_0,p_1}$ is defined to be the space of elements $x \in X_0 + X_1$ such that*

$$x = \sum_{\in\mathbb{M}} \varphi(1,u_m)x_m \quad (\textit{convergence in } X_0 + X_1),$$

where $x_m \in X_0 \cap X_1$, $\{\| x_m\|_{X_0}\} \in l_{p_0}$, and $\{u_m \| x_m\|_{X_1}\} \in l_{p_1}$ (see [22]).

In the case of $\varphi(s,t) = s^{1-\theta}t^{\theta}$, $0 < \theta < 1$ these spaces where introduced by J.-L.Lions and J.Peetre and were called the spaces of means. It is easily seen that our generalization of the Lions–Peetre method of means is a particular case of the general method of means considered in [4]. That is why such properties as Banach space property, interpolation property and essential K-monotonicity follow immediately from the definition.

It is important that the definition of $\varphi(X_0, X_1)_{p_0,p_1}$ is independent of the choice of a balanced sequence $\{u_m\}$ for φ. This follows from the next proposition, which gives us the description of $\varphi(X_0, X_1)_{p_0,p_1}$ in terms of the K-functional and the Calderon–Lozanovskii construction as well. Note that this description has nothing common with the description of the Lions–Peetre method of means obtained in [4].

Recall that the Calderon–Lozanovskii construction allows to introduce an intermediate quasi-Banach lattice $\varphi(X_0, X_1)$ with the help of two quasi-Banach lattices X_0 and X_1 and an interpolation function φ. The space $\varphi(X_0, X_1)$ consists of $x \in X_0 + X_1$ such that

$$|x| = \varphi(|x_0|, |x_1|)$$

for some $x_0 \in X_0$ and $x_1 \in X_1$. The quasi-norm is defined by

$$\|x\|_{\varphi(X_0,X_1)} = \inf_{\substack{x_0,\, x_1 \\ \varphi(|x_0|,\,|x_1|) = |x|}} \max(\|x_0\|_{X_0}, \|x_1\|_{X_1}).$$

Recall that interpolation function φ is called non-degenerate if the ranges of the functions $\varphi(s,1)$ and $\varphi(1,t)$ coincide with $(0,\infty)$. Note that in the case of non-degenerate functions the corresponding balanced sequences turns out to be bilateral, i.e., be defined on $\mathbb{Z}$.

Proposition 5.1. *[23] Suppose that $\varphi \in \Phi_0$. If φ is non-degenerate, the space $\varphi(X_0,\, X_1)_{p_0,p_1}$ consists of $x \in X_0 + X_1$ such that*

$$\{K(1, w_m, x; \{X_0,\, X_1\}\} \in \varphi(l_{p_0},\, l_{p_1}(w_m^{-1})),$$

where $\{w_m\}$ is a balanced sequence for the function $K(s,t,x;\{X_0, X_1\})$.

If $\varphi(1,t)$ *is bounded and* $\varphi(s,1)$ *is unbounded, then* $\varphi(X_0,X_1)_{p_0,p_1}$ *consists of* $x \in X_0$ *such that*

$$\{K(1,w_m,x;\{X_0,\,X_1\}\} \in \varphi(l_{p_0},\,l_{p_1}(w_m^{-1})),$$

where $\{w_m\}$ *is a balanced sequence for the function* $K(s,t,x;\{X_0,X_1\})$.

If $\varphi(1,t)$ *is unbounded and* $\varphi(s,1)$ *is bounded, then* $\varphi(X_0,X_1)_{p_0,p_1}$ *consists of* $x \in X_1$ *such that*

$$\{K(1,w_m,x;\{X_0,\,X_1\}\} \in \varphi(l_{p_0},\,l_{p_1}(w_m^{-1})),$$

where $\{w_m\}$ *is a balanced sequence for the function* $K(s,t,x;\{X_0,X_1\})$.

If both $\varphi(1,t)$ *and* $\varphi(s,1)$ *are bounded, then* $\varphi(X_0,X_1)_{p_0,p_1} = X_0 \cap X_1$.

It turns out that this Proposition may be used for definition of $\varphi(X_0,X_1)_{p_0,p_1}$ if $0 < p_0, p_1 \leqslant \infty$.

Definition 5.4. *[24] Suppose that* $\{X_0,X_1\}$ *is a Banach couple and* $0 < p_0,p_1 \leqslant \infty$, $\varphi \in \Phi_0$. *If* φ *is non-degenerate, then* $\varphi(X_0,X_1)_{p_0,p_1}$ *is defined to consist of* $x \in X_0 + X_1$ *such that*

$$\{K(1,w_m,x;\{X_0,\,X_1\}\} \in \varphi(l_{p_0},\,l_{p_1}(w_m^{-1})), \tag{5.1}$$

where $\{w_m\}$ *is a balanced sequence for the function* $K(s,t,x;\{X_0,X_1\})$.

If $\varphi(1,t)$ *is bounded and* $\varphi(s,1)$ *is unbounded, then* $\varphi(X_0,X_1)_{p_0,p_1}$ *is defined to consist of* $x \in X_0$ *such that*

$$\{K(1,w_m,x;\{X_0,\,X_1\}\} \in \varphi(l_{p_0},\,l_{p_1}(w_m^{-1})),$$

where $\{w_m\}$ *is the same sequence.*

If $\varphi(1,t)$ *is unbounded and* $\varphi(s,1)$ *is bounded, then* $\varphi(X_0,X_1)_{p_0,p_1}$ *is defined to consist of* $x \in X_1$ *such that*

$$\{K(1,w_m,x;\{X_0,\,X_1\}\} \in \varphi(l_{p_0},\,l_{p_1}(w_m^{-1})),$$

where $\{w_m\}$ *is the same sequence.*

If both $\varphi(1,t)$ *and* $\varphi(s,1)$ *are bounded, then* $\varphi(X_0,X_1)_{p_0,p_1} = X_0 \cap X_1$.

If $\overline{X}$ is a Hilbert couple and $\varphi \in \Phi_0$, then the condition (5.1) on the K-functional is sufficient to $x \in \varphi(H_0,H_1)_{p_0,p_1}$ for all $0 < p_0,p_1 \leqslant \infty$.

If $\varphi(s,t)$ is an interpolation function, then $\varphi^\star(s,t) = 1/\varphi(1/s,\,1/t)$ is also an interpolation function. Denote by $\varphi(H_0,\,H_1)$ the space, which corresponds to the space $l_2(\varphi^\star(1,\,2^{-n})G_n)$ if we identify the couple $\{H_0,\,H_1\}$ with the couple of vector-valued sequence spaces $\{l_2(G_n),\,l_2(2^{-n}G_n)\}$ (see Section 3).

We use in the next subsection that $\varphi\,(H_0,\,H_1) = \varphi(H_0,\,H_1)_{2,2}$ (see [22],[23]).

Recall that $\Lambda_\varphi(H_0,\,H_1)$ denotes the so called generalized Lorentz spaces, corresponding to the interpolation function φ. (It can be defined in a lot of ways (see [7]), for instance as the smallest interpolation space E between H_0 and H_1 such that

$$\|x\|_E \leqslant \varphi\,(\|x\|_{H_0},\,\|x\|_{H_1}).$$

The identity $\Lambda_{\varphi^\star}(H_0,\,H_1) = \varphi(H_0,H_1)_{1,1}$ follows from the definitions.

The generalized Marcinkiewicz spaces $M_\varphi(H_0,\,H_1)$ can be defined through duality to the Lorentz spaces, but it is much more easier to define $M_\varphi(H_0,\,H_1)$ by the K-method. Namely

$$M_\varphi(H_0,\,H_1) = \{x \in H_0 + H_1;\ \sup_{0<s,t} \frac{K(s,\,t,\,x;\,\{H_0,\,H_1\})}{\varphi(s,\,t)} < \infty\}$$

with the natural norm

$$\|x\|_{M_\varphi(H_0,H_1)} = \sup_{0<s,t} \frac{K(s,\,t,\,x;\,\{H_0,\,H_1\})}{\varphi(s,\,t)}.$$

(see [7]).

We easily have $M_\varphi(H_0,\,H_1) = \varphi(H_0,H_1)_{\infty,\infty}$.

5.2 Description of interpolation orbits

The final results on description of interpolation orbits with respect to the Neumann–Schatten operator ideal are obtained in [24].

Recall that for any regular couple $\{H_0, H_1\}$ and any $a \in H_0+H_1$ we have $K(s,t,a,\{H_0,H_1\}) \in \Phi_0$.

Theorem 5.1. *Let $\{H_0, H_1\}$ and $\{F_0, F_1\}$ be regular Hilbert couples, $a \in H_0+H_1$. Then for any $0<p_0,p_1<\infty$*

$$Orb\,(a,\, S_{p_0}(H_0 \to F_0) \cap S_{p_1}(H_1 \to F_1)) = \varphi(F_0,\, F_1)_{p_0,p_1},$$

where $\varphi(s,\, t) = K(s,\, t, a;\, \{H_0,\, H_1\})$.

For any $0<p_0<\infty$

$$Orb\,(a,\, S_{p_0}(H_0 \to F_0) \cap L(H_1 \to F_1)) = \varphi(F_0,\, F_1)_{p_0,\infty},$$

$$Orb\,(a, L(H_0 \to F_0) \cap L(H_1 \to F_1)) = \varphi(F_0,\, F_1)_{\infty,\infty}.$$

The latter identity actually coincides with the Sedaev theorem [29].

Corollary 5.1. *Let $\varphi \in \Phi_0$ and $0 < p_0, p_1, s_0, s_1 \leqslant \infty$, then for any $T \in S_{p_0}(H_0 \to F_0) \cap S_{p_1}(H_1 \to F_0)$ we have $T : \varphi(H_0, H_1)_{s_0,s_1} \to \varphi(F_0, F_1)_{t_0,t_1}$, where $1/t_0 = 1/s_0 + 1/p_0$, $1/t_0 = 1/s_0 + 1/p_0$.*

Corollary 5.2. *For any $a \in H_0+H_1$ and for any $0<p_0,p_1 \leqslant \infty$*

$$Orb\,(a,\, S_{p_0}(H_0 \to F_0) \cap S_{p_1}(H_1 \to F_1)) = \varphi(F_0,\, F_1)^{\circ}_{p_0,p_1},$$

where $\varphi(s,\, t) = K(s,\, t, a;\, \{H_0,\, H_1\})$.

Recall that X° denotes the closure of $F_0 \cap F_1$ in X.

Corollary 5.3. *For any $a \in H_0+H_1$*

$$Orb\,(a,\, L(H_0 \to F_0) \cap L(H_1 \to F_1)) = M_{\varphi}(F_0,\, F_1),$$

$$Orb\,(a,\, S_{\infty}(H_0 \to F_0) \cap S_{\infty}(H_1 \to F_1)) = M^{o}_{\varphi}(F_0,\, F_1),$$

$$Orb\,(a,\, S_2(H_0 \to F_0) \cap S_2(H_1 \to F_1) = \varphi(F_0,\, F_1),$$

$$Orb\,(a,\, S_1(H_0 \to F_0) \cap S_1(H_1 \to F_1) = \Lambda_{\varphi^{\star}}(F_0,\, F_1),$$

where $\varphi(s,\, t) = K(s,\, t, a;\, \{H_0,\, H_1\})$.

6. THE COMPLEX METHOD AND EMBEDDING FOR CROSS-NORM IDEALS

The results described in previous Section 4 can be naturally interpreted as embedding theorems for operator spaces. For instance Theorem 4.1 is an embedding

$$S_{p_0}(H_0 \to F_0) \cap S_{p_1}(H_1 \to F_1) \subset L(\overline{H}_{\theta,r} \to \overline{F}_{\theta,q}).$$

We are going to consider the analogous embedding for more sophisticated constructions than the intersection of spaces.

Note that if a couple $\overline{H}$ is regular then $S_{p_0}(H_0 \to F_0)$ and $S_{p_1}(H_1 \to F_1)$ form a Banach couple.

We begin with the complex method by A.P.Calderon, applied to $\{S_{p_0}(H_0),\, S_{p_1}(H_1)\}$. This situation was considered in [31], [15], [19].

Let H_θ, $(0 < \theta < 1)$, be the unique Hilbert scale connecting H_0 and H_1. If $H_0 = l_2(G_n)$ and $H_1 = l_2(2^{-n}G_n)$, then $H_\theta = l_2(2^{-n\theta}G_n)$.

The simplest approach is to represent the couple $\{S_{p_0}(H_0),\, S_{p_1}(H_1)\}$ as a couple of weighted matrix spaces (see Section 3)

$$\{S_{p_0}(l_2(G_n)),\, S_{p_1}(l_2(G_n))(2^{j-i})\}. \tag{6.1}$$

The complex method, applied to the couple $\{S_{p_0}(H),\, S_{p_1}(H)\}$, where $H_0 = H_1 = H$ is well studied and we have (see [8], [34], [32])

$$[S_{p_0}(H),\, S_{p_1}(H)]_\theta \;=\; S_{p_\theta}(H),$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$. Therefore (see [19]),

$$[S_{p_0}(l_2(G_n)),\, S_{p_1}(l_2(G_n))(2^{j-i})]_\theta = S_{p_\theta}(l_2(G_n))(2^{\theta(j-i)})$$

for any $0 < \theta < 1$. Thus we obtain the following

Theorem 6.1. *For any regular Hilbert couple $\{H_0,\, H_1\}$ and $1 \leqslant p_0,\, p_1 \leqslant \infty$*

$$[S_{p_0}(H_0),\, S_{p_1}(H_1)]_\theta \;=\; S_{p_\theta}(H_\theta). \tag{6.2}$$

In the case of the couple $\{L(H_0),\, L(H_1)\}$ we obtain

Theorem 6.2. *For any regular Hilbert couple $\{H_0,\, H_1\}$*

$$[L(H_0),\, L(H_1)]^\theta \;=\; L(H_\theta),$$

where $[\,\cdot\,,\,\cdot\,]^\theta$ *is the second Calderon complex method.*

By the way (6.2) implies that

$$S_{p_0}(H_0) \cap S_{p_1}(H_1) \subset S_{p_\theta}(H_\theta). \tag{6.3}$$

Moreover, $S_{p_\theta}(H_\theta)$ is an intermediate space between $S_{p_0}(H_0)$ and $S_{p_1}(H_1)$, i.e.,

$$S_{p_0}(H_0) \cap S_{p_1}(H_1) \subset S_{p_\theta}(H_\theta) \subset S_{p_0}(H_0) \,+\, S_{p_1}(H_1).$$

We also have

$$\|T\|_{S_{p_\theta}(H_\theta)} \;\leqslant\; \|T\|^{1-\theta}_{S_{p_0}(H_0)} \|T\|^{\theta}_{S_{p_1}(H_1)},$$

for any $S_{p_0}(H_0) \cap S_{p_1}(H_1)$, which follows from (6.2). This result was obtained first in [28].

It seems natural to suppose that $\Phi(H_\theta)$ are the most typical examples of interpolation space and trivially are examples of intermediate spaces. However as we see from the following theorem that this is not true for majority of Hilbert couples.

Theorem 6.3. *Let $\overline{H} \,=\, \{H_0,\, H_1\}$ be generalized spectrally filling couple. Suppose that Φ is a cross-norm ideal. Then $\Phi(H_\theta)$ is an intermediate space between $S_{p_0}(H_0)$ and $S_{p_1}(H_1)$, if and only if $\Phi = S_{p_\theta}$.*

This theorem contradicts to the results of [10] that one-sided interpolation takes place for any Hilbert couple and for any the Neumann-Schatten ideals including nuclear and the Hilbert-Schmidt ideals. In particular it was alleged that

$$S_1(H_0 \to F_0) \cap L(H_1 \to F_1) \subset S_1(H_\theta \to F_\theta), \tag{6.4}$$

what is impossible for generalized spectrally filling couples in view of Theorem 6.3.

However the embedding similar to (6.4) are possible and not only for trivial couples.

Denote by $\mu_n = 2^n$ for $n \geqslant 0$, and consider the Hilbert couple $\{l_2,\, l_2(2^{\mu_n})\}$, which has extremely sparse spectrum. Then

$$\Phi_0(H_0 \to H_0) \cap \Phi_1(H_1 \to H_1) \subset (\Phi_0 \cap \Phi_1)(H_\theta)$$

for any $0 < \theta < 1$ and any cross-norm ideals Φ_0, Φ_1. Thus one-sided results take place for this couple.

Finally recall that one-sided results for compact operators take place, i.e.,

$$S_\infty(H_0 \to F_0) \cap L(H_1 \to F_1) \subset S_\infty(H_\theta \to F_\theta)$$

for any Hilbert couples and $0 < \theta < 1$ (see [6] for full references).

7. DESCRIPTION OF THE SUM AND THE INTERSECTION OF OPERATOR SPACES OF HILBERT COUPLES

7.1 Weighted space description

The simplest interpolation spaces for $\overline{X} = \{X_0,\, X_1\}$ are the sum and the intersection of the spaces X_0, X_1. More or less appropriate description of the spaces

$$S_{p_0}(H_0) \cap S_{p_1}(H_1) \quad \text{and} \quad S_{p_0}(H_0) \,+\, S_{p_1}(H_1)$$

for $p_0 \neq p_1$ are unknown. We present here some facts for the case $p_0 = p_1$. In the case $p_0 = p_1 = \infty$ we consider the couple $\{L(H_0),\, L(H_1)\}$ (see [20], [21]).

Again we use the representation of $\{H_0,\, H_1\}$ as a couple of sequence spaces $\{l_2(G_n),\, l_2(2^{-n}G_n)\}$.

Consider the space $\mathfrak{M}$ of arbitrary matrices $\{T_{ij}\}$, where $i,\, j \in \mathbb{Z}$ and $T_{ij} \in L(G_j \to G_i)$. Let us introduce $\mathcal{P}^k_+$, $\mathcal{P}^k_-$, $\mathcal{P}^k_0$, mapping this space $\mathfrak{M}$.

We put $\mathcal{P}^k_+(T)_{ij} = T_{ij}$ for $i < j+k$ and $\mathcal{P}^k_+(T)_{ij} = 0$ for $i \geqslant j+k$; $\mathcal{P}^k_-(T)_{ij} = T_{ij}$, for $i > j+k$ and $\mathcal{P}^k_-(T)_{ij} = 0$ for $i \geqslant j+k$; $\mathcal{P}^k_0(T)_{ij} = T_{ij}$, for $i = j+k$ and $\mathcal{P}^k_0(T)_{ij} = 0$ for $i \neq j+k$.

Thus $\mathcal{P}^k_- + \mathcal{P}^k_+ + \mathcal{P}^k_0 = I$ for any $k \in \mathbb{Z}$.

These maps $\mathcal{P}^k_\pm$ are triangular truncations. It's properties are well studied in $S_p(H)$ (see [1]). Their main property is the boundedness in any S_p for $1 < p < \infty$. It is also known that they are unbounded on the space of all bounded linear operators and the space of nuclear operators.

The maps $\mathcal{P}^k_0$ are diagonal truncations and they are bounded in every S_p for $1 \leqslant p \leqslant \infty$, and in the space of all bounded linear operators. Moreover,

$$\|\mathcal{P}^k_0(T)\|_{S_p} \leqslant \|T\|_{S_p},$$

for all matrices T and $1 \leqslant p \leqslant \infty$.

Let us denote by $\mathcal{H}$ an analog of the Hilbert transform $\mathcal{H} = \mathcal{P}^o_+ - \mathcal{P}^o_-$, and introduce the spaces

$$H^p(l_2(G_n)) \,=\, S_p(l_2(G_n)) \cap \mathcal{H}(S_p(l_2(G_n))),$$

for any $1 \leqslant p < \infty$, and

$$H^\infty(l_2(G_n)) \,=\, L(l_2(G_n)) \cap \mathcal{H}(L(l_2(G_n))).$$

The boundedness of $\mathcal{H}$ in S_p implies $H^p(l_2(G_n)) \,=\, S_p(l_2(G_n))$ for $1 < p < \infty$.

Let us denote also

$$BMO(l_2(G_n)) \,=\, L(l_2(G_n)) \,+\, \mathcal{H}(L(l_2(G_n))).$$

It is easily seen that for the definition of these spaces we can use the projections $\mathcal{P}^k_\pm$ instead of $\mathcal{P}^o_\pm$. Thus
$$BMO(l_2(G_n)) \;=\; L(l_2(G_n)) \,+\, \mathcal{H}_k(L(l_2(G_n))),$$
where $\mathcal{H}_k = \mathcal{P}^k_+ - \mathcal{P}^k_-$, but these definitions lead to a family of equivalent norms on BMO
$$\|T\|_{BMO_k(l_2(G_n))} = \inf_{\substack{U_0,\,U_1\\ T = U_1 + \mathcal{H}_k(U_2)}} \|U_1\|_{L(l_2(G_n))} \,+\, \|U_2\|_{L(l_2(G_n))}.$$

The spaces, which were introduced, are useful for the description of the intersection and the sum of operator spaces.

Theorem 7.1. *For any* $1 < p < \infty$
$$S_p(H_0) \cap S_p(H_1) \;=\; S_p(H_0)(\max(1,\, 2^{j-i})),$$
$$S_p(H_0) \,+\, S_p(H_1) \;=\; S_p(H_0)(\min(1,\, 2^{j-i})).$$

Since the space H_0 is identified with $l_2(G_n)$ and the space $S_p(H_0)$ with some space of matrices, we denote by $S_p(H_0)(w_{ij})$ the space of matrices $T_{ij} \in L(G_j \to G_i)$ such that the matrix $\{w_{ij}T_{ij}\} \in S_p(l_2(G_n))$.

For the description of the sum and the intersection in the extreme cases $p = 1, \infty$ we have to use the spaces introduced above.

Theorem 7.2. *We have*
$$S_1(H_0) \cap S_1(H_1) = H^1(l_2(G_n))(\max(1,\, 2^{j-i})),$$
$$L(H_0) \cap L(H_1) = H^\infty(l_2(G_n))(\max(1,\, 2^{j-i})),$$
$$L(H_0) \,+\, L(H_1) = BMO(l_2(G_n))(\min(1,\, 2^{j-i})).$$

The description of spaces from Theorem 7.2 can be used to settle the question posed by L.Maligranda (see [14]) concerning general properties of the complex method. The question was whether the space
$$[E_0 \cap E_1,\, E_0 + E_1]_{1/2}$$
is equal to $[E_0,\, E_1]_{1/2}$. It is known that these two spaces are equal for couples $\{E_0, E_1\}$ of Banach lattices, for example for Hilbert couples. However, this is not so for the couple $\{L(H_0),\, L(H_1)\}$, if $\{H_0,\, H_1\}$ is spectrally filling (see [25] for more examples). Let us outline that
$$[L(H_0) \cap L(H_1),\, L(H_0) \,+\, L(H_1)]^{1/2} \neq [L(H_0),\, L(H_1)]^{1/2}.$$
We have already seen in Section 6
$$[L(H_0),\, L(H_1)]^{1/2} \;=\; L(H_{1/2}).$$
From the other side the description above yields
$$[L(H_0) \cap L(H_1),\, L(H_0) \,+\, L(H_1)]^{1/2} = [H^\infty,\, BMO]^{1/2}(H_{1/2}).$$

The spaces $[H^\infty,\, BMO]^{1/2}(H_{1/2})$ and $L(H_{1/2})$ don't coincide because the first is invariant with respect to $\mathcal{H}$, but the second is not.

The estimation of the K-functional for couples of cross-norm ideals is also possible.

It turns out that
$$K(t,\, T;\, \{S_p(H_0),\, S_p(H_1)\}) \asymp \|T_{ij}\min(1,\, 2^{j-i}t)\|_{S_p(l_2(G_n))}$$
for $1 < p < \infty$.

The corresponding result for $p = \infty$ is not so "smooth"
$$K(2^k,\, T;\, \{L(H_0),\, L(H_1)\}) \asymp \|T_{ij}\min(1,\, 2^{k+j-i})\|_{BMO_k(H_0)}$$
since the norms $\|T\|_{BMO_k}$ themselves are defined again with the g.l.b.

7.2 Alternative descriptions of $L(H_0) + L(H_1)$

Let us denote by E_m, where $m \in \mathbb{Z}$, the orthogonal projection onto the linear span of G_j, $(j \leqslant m)$ in $l_2(G_n)$ and in $l_2(2^{-n}G_n)$. Evidently $E_m : \{l_2(G_n), l_2(2^{-n}G_n)\} \to \{l_2(G_n), l_2(2^{-n}G_n)\}$.

Theorem 7.3. *[2] The space $L(H_0) + L(H_1)$ consists of $T \in L(H_0 \cap H_1 \to H_0 + H_1)$ such that*

$$\sup_{n\in\mathbb{Z}} \|E_n T(I - E_n)\|_{L(H_0)} + \sup_{n\in\mathbb{Z}} \|(I - E_n)TE_n\|_{L(H_1)} < \infty.$$

For any $T \in L(H_0) + L(H_1)$

$$K(2^p, T, \{L(H_0), L(H_1)\}) \asymp \sup_{n\in\mathbb{Z}} \|E_{n+p}T(I - E_n)\|_{L(H_0)} + 2^p \sup_{n\in\mathbb{Z}} \|(I - E_{n+p})TE_n\|_{L(H_1)}.$$

Theorem 7.4. *[27] The space $L(H_0) + L(H_1)$ consists of $T \in L(H_0 \cap H_1 \to H_0 + H_1)$ such that*

$$\sup_{x\in H_0\cap H_1} K(1/\|x\|_{H_0}, 1/\|x\|_{H_1}, T(x), \{H_0, H_1\})$$

is finite, and for any $T \in L(H_0) + L(H_1)$

$$K(s, t, T, \{L(H_0), L(H_1)\}) \asymp \sup_{x\in H_0\cap H_1} K(s/\|x\|_{H_0}, t/\|x\|_{H_1}, T(x), \{H_0, H_1\}).$$

This theorem is based on the description of the minimal operator ideal mapping Hilbert couple. It is easily seen that the minimal ideal consists of $T \in L(H_0) \cap L(H_1)$, such that $T = \sum_{j=1}^{\infty} T_j$ where $T_j \in L(H_0) \cap L(H_1)$ are rank-one operators, provided

$$\sum_{j=1}^{\infty} \|T_j\|_{L(H_0)\cap L(H_1)} < \infty.$$

This minimal ideal is also called the ideal of coherently nuclear operators and is denoted by $CohN\{H_0, H_1\}$.

Theorem 7.5. *[27] For any Hilbert couple*

$$CohN\{H_0, H_1\} = S_1(H_0) \cap S_1(H_1).$$

8. THE LIONS-PEETRE CONSTRUCTION FOR COUPLES OF NEUMANN-SCHATTEN IDEALS

We can see from Theorem 6.3 that for the description of interpolation spaces, which are not complex method spaces, we need to introduce some new classes of operator spaces.

Let a Hilbert couple $\{H_0, H_1\}$ be identified with the couple of vector-valued sequence spaces

$$\{l_2(G_n), l_2(2^{-n}G_n)\}.$$

Let $\theta \in \mathbb{R},\ 0 < p < \infty,\ 0 < q \leqslant \infty$. We introduce the matrix space

$$M_{pq}(H_\theta) = \{T_{ij} : \sum_{k=-\infty}^{\infty} (2^{k\theta}(\sum_{\substack{j = -\infty, \\ k = j - i}}^{\infty} \|T_{ij}\|^p_{S_p(G_j\to G_i)})^{1/p})^q < \infty\}$$

with natural corrections in case of p or $q = \infty$.

For brevity denote $M_{pp}(H_\theta) = M_p(H_\theta)$. These matrix spaces are close to the Neumann-Schatten ideals.

Indeed, $M_2(H_\theta) = S_2(H_\theta)$ for any θ. For $p \neq 2$ there are simple inclusion maps $M_p(H_\theta) \subset S_p(H_\theta)$, if $1 \leqslant p \leqslant 2$, and $S_p(H_\theta) \subset M_p(H_\theta)$, if $2 \leqslant p \leqslant \infty$ and any θ.

The interaction of scales $M_p(H_\theta)$ and $S_p(H_\theta)$ turns out to be deeper (see [21]).

Lemma 8.1. *For any* $\theta_0 \neq \theta_1 \in \mathbb{R}$

$$M_{p_0}(H_{\theta_0}) \cap M_{p_1}(H_{\theta_1}) \subset S_p(H_\theta),$$

where $\theta = (1-\mu)\theta_0 + \mu\theta_1,\ 1/p = (1-\mu)/p_0 + \mu/p_1$ *for* $0 < \mu < 1$, *and*

$$\|T\|_{S_p(H_\theta)} \leqslant C\, \|T\|^{1-\mu}_{M_{p_0}(H_{\theta_0})} \cdot \|T\|^{\mu}_{M_{p_1}(H_{\theta_1})}.$$

This lemma implies that the reiteration theorem for the Lions-Peetre construction can be applied, and we obtain

Theorem 8.1. *For any* $\theta_0 \neq \theta_1$ *and* $1 \leqslant p_0,\, p_1 \leqslant \infty$

$$(S_{p_0}(H_{\theta_0}), S_{p_1}(H_{\theta_1}))_{\eta,q} = (M_{p_0}(H_{\theta_0}), M_{p_1}(H_{\theta_1}))_{\eta,q},$$

where $0 < \eta < 1,\ 0 < q \leqslant \infty$.

Theorem 8.2. *For any* $\theta_0 \neq \theta_1$ *and* $1 \leqslant p,\, p_0,\, p_1 \leqslant \infty$

$$(S_p(H_{\theta_0}), S_p(H_{\theta_1}))_{\eta,q} = M_{pq}(H_\theta),$$

where $\theta = (1-\eta)\theta + \eta\theta_1$ *and any* $0 < q \leqslant \infty$. *If* $1/q = (1-\eta)/p_0 + \eta/p_1$, *then*

$$(S_{p_0}(H_{\theta_0}),\, S_{p_1}(H_{\theta_1}))_{\eta,q} = M_q(H_\theta).$$

The spaces $(S_{p_0}(H_{\theta_0}),\, S_{p_1}(H_{\theta_1}))_{\eta,q}$ can be also described for ”non-diagonal” cases, i.e.,

$$1/q \neq (1-\eta)/p_0 + \eta/p_1,$$

but it demands a consideration of non-commutative Lorentz spaces L_{pq} on a gauge space with a measure on projections (see [30], [21]).

Note that for some p the matrix spaces $M_p(H_\theta)$ have an operator description.

Indeed,

$$M_\infty(H_\theta) = L(l_1(2^{-n\theta}G_n) \to l_\infty(2^{-n\theta}G_n)).$$

Recall (see [3]) that

$$(l_2(G_n),\, l_2(2^{-n}G_n))_{\theta,p} = l_p(2^{-n\theta}G_n),$$

therefore

$$M_\infty(H_\theta)\ =\ L((H_0,\, H_1)_{\theta,1} \to (H_0,\, H_1)_{\theta,\infty}),$$

and we obtain

$$(L(H_0),\, L(H_1))_{\theta,\infty} = L((H_0,\, H_1)_{\theta,1} \to (H_0,\, H_1)_{\theta,\infty})$$

by Theorem 8.2.

Analogously, since

$$M_1(H_\theta) = \{T_{ij} : \sum_{i,j\in\mathbb{Z}} 2^{(j-i)\theta}\|T_{ij}\|_{S_1(G_j\to G_i)} < \infty\}$$

we obtain

$$M_1(H_\theta)\ =\ N(c_0(2^{-n\theta}G_n) \to l_1(2^{-n\theta}G_n)),$$

where $N(X \to Y)$ is the space of all nuclear operators mapping X to Y. Hence

$$(S_1(H_0),\, S_1(H_1))_{\theta,1} = N((H_0,\, H_1)^o_{\theta,\infty} \to (H_0,\, H_1)_{\theta,1}).$$

The equality

$$(S_1(H_0),\, S_1(H_1))_{\theta,1} \;=\; M_1(H_\theta)$$

implies the embedding

$$S_1(H_0) \cap S_1(H_1) \subset N((H_0,\, H_1)_{\theta,\infty} \to (H_0,\, H_1)_{\theta,1}),$$

which slightly improves the previous result in this direction (see Theorem 4.1), which said that

$$S_1(H_0) \cap S_1(H_1) \subset L((H_0,\, H_1)_{\theta,\infty} \to (H_0,\, H_1)_{\theta,1}).$$

9. INTERPOLATION OF IDEAL PROPERTIES IN INTERPOLATION HILBERT SPACES

In this Section we discuss two-sided interpolation of cross-norm ideals in general interpolation Hilbert space for a Hilbert couple.

Recall that if a Hilbert couple is represented in the form $\{l_2(G_n),\, l_2(2^{-n}G_n)\}$ then any interpolation Hilbert space between H_0 and H_1 can be identified with the space $l_2(\varphi(1,\, 2^{-n})G_n)$, where $\varphi(s,t)$ is an interpolation function. If also note that $T \in S_2(l_2(w_n G_n))$ is equivalent to

$$\sum_{i,j\in\mathbb{Z}} \|w_i w_j^{-1} T_{ij}\|^2_{S_2(G_j \to G_i)} < \infty,$$

then the following interpolation theorem becomes almost evident (see [18]).

Theorem 9.1. *If $T \in S_2(H_0) \cap S_2(H_1)$, then $T \in S_2(H)$, where H is an arbitrary interpolation Hilbert space between H_0 and H_1.*

The factorization Theorem 4.3 and Theorem 9.1 imply

Theorem 9.2. *If $T \in S_1(H_0) \cap S_1(H_1)$, then $T \in S_1(H)$, where H is an arbitrary interpolation Hilbert space between H_0 and H_1.*

The interpolation of arbitrary ideal property in any interpolation Hilbert space is not established yet, but for a rather large class of interpolation spaces it is already done.

Recall that interpolation space H is called R-interpolation space, where $R(s,\, t)$ is a function of two positive variables, if for any $T \in L(H_0) \cap L(H_1)$ we have

$$\|T\|_{L(H\to H)} \leqslant R\,(\,\|T\|_{L(H_0\to H_0)},\, \|T\|_{L(H_1\to H_1)}).$$

Definition 9.1. *If the function $R(s,\, t) \to 0$ as $s \to 0$ for any t, and $R(s,\, t) \to 0$ as $t \to 0$ for any s, then R-interpolation space H is called a proper interpolation space between H_0 and H_1.*

The proper interpolation spaces are remarkable by their property of one-sided interpolation of compactness (see [6]), i.e., we have

Theorem 9.3. *If $T \in S_\infty(H_0) \cap L(H_1)$ or $T \in S_\infty(H_1) \cap L(H_0)$, then $T \in S_\infty(H)$, where H is any proper interpolation Hilbert space.*

For interpolation ideals (see Section 2) we have the following

Theorem 9.4. *If $T \in J(H_0) \cap J(H_1)$, where J is any interpolation ideal, then $T \in J(H)$ for any proper interpolation space H between H_0 and H_1.*

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