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EXISTENCE AND UNIQUENESS RESULTS FOR A COUPLED PROBLEM IN CONTINUUM THERMOMECHANICS

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Abstract: this paper presents results on solvability of multidimensional systems of equations of thermoviscoelasticity. Both compressible and incompressible continua are considered. The existence and uniqueness of regular, weak and weak-renormalized solutions are given, both local and global. The systems under consideration are coupled systems of a motion equation and an energy equation. The results are based on the reduction of the system of equations of thermoviscoelasticity to an operator equation in the suitable Banach space. The operator equation is constructed by means of successive solving of the motion equation and the energy equation. The corresponding apriori estimates admit to obtain the solvability of operator equations by means of application of various fix-point theorems. The theory of anisotropic Sobolev spaces with a mixed norm is used.

Key words and phrases: Thermoviscoelastic continuum, successive approximations, a priori estimates, fix-point theorem, weak solution, weak-renormalized solution, Oberbeck-Boussinesq type system.

НЕКОТОРЫЕ РЕЗУЛЬТАТЫ О СУЩЕСТВОВАНИИ И ЕДИНСТВЕННОСТИ ДЛЯ СВЯЗАННЫХ ЗАДАЧ ТЕРМОМЕХАНИКИ

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Аннотация: в этой статье представлены результаты о разрешимости многомерных систем уравнений термовязкоупругости. Рассматриваются как сжимаемые так и несжимаемые среды. Приводятся результаты о существовании и единственности сильных, слабых и слабо-ренормализованных решений, как локальные, так и глобальные. Рассматриваемые системы уравнений являются связанными системами уравнений движения и уравнения сохранения энергии. Основным методом получения результатов является сведение исходной системы уравнений термовязкоупругости к операторному уравнению в подходящем банаховом пространстве, основанное на последовательном решении уравнений движения и уравнения сохранения энергии. Соответствующие априорные оценки позволяют получить разрешимость операторных уравнений с помощью применения различных теорем о неподвижной точке. Используется теория анизотропных пространств Соболева со смешанной нормой.

Ключевые слова: термовязкоупругая среда, последовательные приближения, априорные оценки, теория неподвижных точек, слабое решение, слабо-ренормализованное решение, системы типа Обербека-Буссинеска.

1. INTRODUCTION

In the present paper we review existence results for multidimensional thermoviscoelastic systems. The systems under consideration describe heat conductive materials which have the properties both of elasticity and viscosity.

Thermoviscoelastic system represents balance laws for the linear momentum and energy. The balance laws of momentum and energy ultimately lead to the nonlinear coupled system of thermoviscoelasticity.

The linear momentum balance equation is specified by the stress tensor given by a rheology law of the appropriate type. The energy equation is governed by thermodynamic potentials (the free energy, the internal energy, the dissipation potential et.c.) which characterize the material. Different assumptions about the form of the rheology law and the thermodynamic potentials generate a variety of system of thermoviscoelasticity.

The existence of solutions to such systems is established by a suitable successive approximation method to regularized (if needed) system in a suitable functional space, proof of solvability of which is based on appropriately chosen successive approximations, global a priori estimates, application of a fixed point theorem and pass to the limit.

In this paper we present the series of results on multidimensional mathematical models of thermoviscoelasticity. The subject matter is largely defined by authors' own research interests. The review does not claim to be exhaustive in its domain and is only intended to demonstrate wildly used applications of the based on fix-point arguments methods in the theory of thermoviscoelasticity.

The paper consists of five sections. In Section 2-3 we introduce fundamental concepts and basic notions used for description of the dynamics of a thermoviscoelastic continua. In Section 4 we review the known results for compressible thermoviscoelastic continua. In Section 5 the known results for incompressible thermoviscoelastic continuum are presented.

2. FUNDAMENTAL CONCEPTS

2.1. Dynamics of continuum

(e. g. [72], Sect.1, [43], Sect. 1.1–1.5.) The object of study is a continum filling a bounded volume $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ with the boundary $\partial\Omega$, which is supposed to be sufficiently smooth. The particle occupying the coordinate $X \in \Omega$ at the moment of time $t = 0$ is identified with this coordinate. To describe the medium it suffices to know the position at time $0 \leqslant t \leqslant T < +\infty$ of any particle X, i. e. the function $x = u(t, X)$. It is supposed that for any t the transformation $x = u(t, X)$ is sufficiently smooth and has the inverse $X = U(t, x)$. Set $\Omega_0 = \Omega$ and $\Omega_t = u(t, \Omega)$ for $t > 0$; then $x = u(t, X)$ is a one-to-one mapping of Ω_0 onto Ω_t and $X = U(t, x)$ is a one-to-one mapping of Ω_t onto Ω_0 and the inverse $x = u(t, X)$.

The velocity and the acceleration of a particle X at time t are respectively defined as follows:

$$v(t, X) = \partial u(t, X)/\partial t, \quad w(t, X) = \partial^2 u(t, X)/\partial t^2. \tag{2.1}$$

The motion description of a continuum in terms of functions $x = u(t, X)$ is associated with the name of Lagrange (so-called Lagrangian description).

There exists another approach.

By $V(t, x)$ denote the velocity of the particle X occupying the point x at time t, so that

$$V(t, x) = v(t, U(t, x)). \tag{2.2}$$

The function $V(t, x)$ defined on $Q_T = \{(t, x)\colon 0 \leqslant t \leqslant T; x \in \Omega_t\}$ is called the *velocity field.*

The Cauchy problem

$$du(t,x)/dt = V(t,u(t,X)),\ u(0,X) = X \tag{2.3}$$

(provided that it is uniquely solvable) demonstrates that knowing $V(t,x)$ we can uniquely determine the function $x = u(t,X)$, i. e. the trajectory of the particle X.

Conversely, (2.2) allows to derive the velocity field $U(t,x)$ knowing the function $x = u(t,X)$.

Thus it suffices to know $U(t,x)$ in order to describe the motion of a continuum. This approach is associated with Euler.

The variables X and x are called Lagrangian and Eulerian coordinates, respectively. When Lagrangian coordinates are used, u is the unknown; in the case of Eulerian coordinates V is the unknown.

Parameters of a continuum may be conveniently expressed either in Lagrangian coordinates (e. g. the stress tensor) or in Eulerian ones (e. g. the strain velocity tensor).

In case of considerable deformations elastic and viscous terms must be taken into account simultaneously. This results in cumbersome terms arising in equations.

Eulerian coordinates are convenient for some conservation laws, while Lagrangian ones are more suitable for others.

To find the function $u(t,X)$ (or $V(t,x)$) one takes into consideration physical laws specific for given kind of continuum.

By $\mathbf{S}(t,x)$ and $F(t,x)$ denote the stress tensor and the external body force respectively (at time t and at point x). An important parameter of a particle X is its density $\rho(t,X)$. The density of the medium at point x is $R(t,x) = \rho(t,U(t,x))$.

The dynamics of a continuum obeys the equation of motion (hereafter the summation convention over the repeated indices is used)

$$R\,(\partial V/\partial t + V_i \partial V/\partial x_i) - \operatorname{Div}\mathbf{S} = RF \tag{2.4}$$

and the continuity equation

$$\partial R/\partial t + \operatorname{div}(RV) = 0. \tag{2.5}$$

Here F is the external body force. The divergence $\operatorname{Div}\mathbf{S}$ of the tensor $\mathbf{S} = (s_{ij})$ is vector $(\operatorname{Div}\mathbf{S})_j$ with components $\partial s_{ij}/\partial x_i$. The system of equations (2.4)–(2.5) is not closed. It can be closed under extra assumptions based on experiments. If $\mathbf{S}$ can be expressed via V, we obtain a closed system of equations, and it makes sense to consider its solvability. Thus, one can assume that $\mathbf{S}$ depends on V. This dependence may be functional or else it can incorporate certain parameters of $u(t,X)$ defined by (2.3).

It may occur that R is predefined for a continuum. For example, if $R \equiv 1$, the continuum is called *incompressible* and equation (2.5) is transformed into

$$\operatorname{div} V = 0. \tag{2.6}$$

The *pressure* of an incompressible continuum is defined by

$$p(t,x) = -\frac{1}{3}\operatorname{tr}\mathbf{S}(t,x), \tag{2.7}$$

and the *stress deviator* is defined by

$$\sigma = \mathbf{S} - pI. \tag{2.8}$$

The motion of an incompressible fluid is governed by the equations

$$\partial V/\partial t + V_i \partial V/\partial x_i - \operatorname{Div}\sigma + \nabla p = F,\ \operatorname{div} V = 0. \tag{2.9}$$

So far we have used Eulerian coordinates in the equations. In Lagrangian coordinates equations of motion (2.4)–(2.5) have the form

$$\partial^2\bar{u}/\partial t^2 - \operatorname{Div}\mathbf{s} = f;\ r = \det|I + \partial\bar{u}/\partial X|\,, \tag{2.10}$$

where $\bar{u}(t,X) = u(t,X) - X$ is the displacement function, $r(t,X) = R(t,u(t,X))$, and $\mathbf{s}(t,X) = \mathbf{S}(t,u(t,X))$.

Various dependencies for $\mathbf{S}$ and R specify classes of corresponding continuum such as ideal fluid, viscous incompressible fluid, elastic medium, or viscoelastic medium. These kinds of continua are subject for further classification.

Real-word problems may involve effects of thermal conductivity. Thermal phenomena require introducing additional parameters and studying the way these parameters affect $\mathbf{S}$, R, and F.

Thus the study of motion becomes entangled with notions belonging to thermodynamics. A continuum is regarded as a set of thermodynamic systems. Specifically, a set of thermodynamic parameters $\{\mu_i\}_{i=1}^n$ is assigned to any particle X. Usually these parameters include the temperature. Some of the parameters can be related to mechanical properties of the continuum and be expressed via u (or V). Other parameters may be of a different nature. A set of parameters is said to form a basis, if the other parameters can be expressed in terms of it.

If thermal phenomena are considered, the number of unknowns increases. One must find not only u (or V), but also μ_i as functions $\mu_i(t,X)$ (or $M_i(t,x) = \mu_i(t,U(t,x))$).

To find $\mu_i(t,X)$ one must derive equations for μ_i, consider the dependence of $\mathbf{S}$, R, and F on μ_i and solve the system obtained in this way. Equations for μ_i are derived from various formulations of the first and the second laws of thermodynamics.

2.2. Fundamental concepts of thermodynamics

(A more detailed exposition may be found e. g. in [43], Chapter 1, Section 1.6.)

It is convenient to use a different notation for the thermodynamic parameters. Specifically, let $(\theta, \{\varepsilon_{ij}\}_{i,j=1}^m)$ stand for $\{\mu_i\}_{i=1}^n$. Thus we single out the temperature θ among the parameters. We mathematically identify a thermodynamic system with the set of parameters $(\theta, \varepsilon_{ij})_{i,j=1}^m$. The *free energy* $f(\theta, \varepsilon_{ij})$ specifies the type of a given thermodynamic system. Any function depending on $(\theta, \varepsilon_{ij})$ is called a *function of state*. Functions of state are numerous. The most important of them are the *entropy* $s(\theta, \varepsilon_{ij})$, the *internal energy* $e(\theta, \varepsilon_{ij})$, and *stresses* $\sigma_{ij}(\theta, \varepsilon_{ij})$. They can be expressed via f as follows:

$$s = -\partial f/\partial\theta;\ \sigma_{ij} = \partial f/\partial\varepsilon_{ij};\ e = f - \theta s. \tag{2.11}$$

In what follows the term ‘thermodynamic system’ refers to a particle X of the medium Ω. The parameters θ, ε_{ij} associated with the particle are considered as functions $\theta(t,X), \varepsilon_{ij}(t,X)$.

It is often the case in applications that the parameters ε_{ij} of the particle X regarded as a thermodynamic system are related to $u(t,X)$ through the formula $\varepsilon(u) = \frac{1}{2}\left(\partial u/\partial X + (\partial u/\partial X)^*\right)$. In this case the matrix $\varepsilon = \varepsilon(u) = (\varepsilon_{ij})_{i,j=1}^m$ is called the *strain tensor*.

The formulas (2.11) show that functions of state can be expressed via f. The same is true if f is replaced by another basic function of state. Then one must obtain an equation for the new basic function of state.

In what follows we use lower-case letters for parameters of a thermodynamic system X and capitals for parameters of the particle situated at point x. Thus, $\theta(t,X)$ is the temperature of the particle X at time t and $\Theta(t,x)$ is the temperature of the particle $X\,(= U(t,x))$ situated at x at time t. Thus we have:

$$\Theta(t,x) = \theta(t,U(t,x)), \quad \theta(t,X) = \Theta(t,u(t,X)). \tag{2.12}$$

Similar equations hold for other functions of state.

A set of functions $\theta(t,X)$, $\varepsilon_{ij}(t,X)$ defining the behaviour of a thermodynamic system X is called a *thermodynamic process.*

The first law of thermodynamics states that if a process is represented by a closed path $\theta(t,X), \varepsilon_{ij}(t,X)$ in the state space $(\theta, \varepsilon_{ij})$, the increment of total energy e equals zero.

This law implies a relation for the increment of internal energy when parameters change:

$$dE = R^{-1}(\varepsilon_{ij}\sigma_{ij}dt + \delta q^e + \delta q^{**}). \tag{2.13}$$

Here the first term is the mechanical energy inflow (it is assumed that the mechanical energy is transformed into internal one), δq^e is the heat inflow, δq^{**} is the inflow of other kinds of energy (except the mechanical and thermal ones) that are transformed into internal energy.

Here dE denotes an exact differential, and δq indicates that the linear form $\delta q = \frac{\partial q}{\partial \varepsilon_{ij}} d\varepsilon_{ij} + \frac{\partial q}{\partial \theta} d\theta$ is not an exact differential.

The exact form of terms δq^e and δq^{**} in the right-hand side of Equation (2.13) is due to 'external' factors with respect to the thermodynamic system.

Thermomechanics studies the behaviour of a continuum Ω assuming that every particle X is a thermodynamic system of mass $r(t,X)$. Thus the continuum is considered as a set of thermodynamic systems. These systems interact, so a change of certain thermodynamic parameters implies a change of other parameters. In particular, this causes energy transmission.

The term δq^e can be made explicit on the basis of the second law of thermodynamics and the entropy balance equation related to this law. This law governs the direction and speed of physical processes.

The *equilibrium process* is a process described by equations not involving speeds of change of state parameters.

Suppose that the equations describing a process relate infinitesimal variations of parameters. If the equations are still satisfied after the signs of all the variations have been changed, the process is called *reversible.*

An equilibrium process can be irreversible (e. g. an equilibrium process of heat transfer in a stationary continuum).

Reversible and irreversible processes can be obtained by combining sequences of equilibrium and nonequilibrium states of the continuum at issue.

The second law of thermodynamics implies

$$\delta Q^e = \theta ds + \delta Q'; \ \delta Q' \geqslant 0. \tag{2.14}$$

Here $\delta Q'$ is called *uncompensated heat.*

If a process is reversible, $\delta Q' = 0$. The converse is not always true. In order to actually use equation (2.14) one needs to know δQ^e and $\delta Q'$. The quantity $\delta Q'$ is often written in the form

$$\delta Q' = \delta q' d\Omega = W^* dt\, d\Omega, \quad W^* dt \geqslant 0, \tag{2.15}$$

where $d\Omega$ is the volume element.

The *dispersionfunction* W^* is determined by physical properties of the continuum. The inequality serves as a criterion showing that relations being used are correct.

Theory of thermomechanical behaviour of continuum postulates the existence of the heat flux vector $q = q(t,x)$ being the quantity of heat transferred in unit time through section of unit area perpendicular to q, so that

$$\delta q^e = -(\operatorname{div} q)\, dt. \tag{2.16}$$

The dependence of q on the parameters of the thermodynamic system varies from one continuum to another. The simplest assumption

$$q = -\varkappa \nabla \theta, \quad \varkappa > 0 \tag{2.17}$$

is called the *Fourier law.*

In the case of the Fourier law we have

$$\delta q^e = \mathrm{div}(\varkappa \nabla \theta) dt. \tag{2.18}$$

Here $\varkappa$ may depend on coordinates, time and various parameters of the thermodynamic system.

It follows from (2.13) that

$$\frac{d}{dt} E = R^{-1}(\sigma_{ij}\varepsilon_{ij} + \mathrm{div}(\varkappa \nabla \theta) + \delta q^{**}/dt). \tag{2.19}$$

Substituting

$$\frac{d}{dt} E(\theta, \varepsilon_{ij}) = \frac{\partial E}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial E}{\partial \varepsilon_{ij}}\frac{d\varepsilon_{ij}}{dt} \tag{2.20}$$

in (2.19), we obtain an equation for θ. Generally, it is nonlinear and parabolic. One must also know the form of the function E and the dependence between ε and σ.

Equations (2.4), (2.5), (2.19), (2.20) and various assumptions about F, R, $\mathbf{S}$, E, and relationships between ε, σ, θ, and q^{**} give rise to various models of thermoviscoelasticity. Note that if Lagrangian coordinates are used in some of the relations and Eulerian ones are used in others, one should ensure a single system for any given equation (e. g. like this is done in (2.12)). If deformations are small, the difference between the Eulerian and Lagrangian coordinates is disregarded. We refrain from spelling the difference between them out, to keep the notation simple.

In the following sections we consider specific models of continua.

2.3. Mathematical models in continuum thermomecanics

In what follows we shall consider continua, the energy equation for which contains only one thermodynamic parameter, namely the temperature θ, and mechanical ones.

Let us dwell on constitutive laws. For incompressible continua often they have a form of differential equation

$$L_1(\sigma) = L_2(D(V)). \tag{2.21}$$

Here L_1 and L_2 are linear differential operators (generally speaking heterogeneous) of orders m and k respectively, $m, k = 0, 1, 2, \ldots$, σ is the stress deviator, $D(V) = (D_{ij})_{i,j=1}^n$, $D_{ij} = \frac{1}{2}(\partial V_i/\partial V_j + \partial V_j/\partial x_i)$, is the strain velocity tensor. The coefficients of L_1 and L_2 may depend on both mechanical and thermodynamic parameters. If (2.21) is solved with respect to σ, then substitution of $\sigma = \sigma(D(V))$ in (2.9) allows to exlude one unknown function σ from (2.9).

For example, the case $m = 0$, $k = 0$ corresponds to the viscous fluid (both Newtonian and non-Newtonian). If $L_1(w) = w$, $L_2(w) = \mathcal{A}_1 w$ and $F = F(\theta)$, then one get the system of Boussinesq's type for thermoviscous incompressible continua (see Section 4 below). The coefficients $\mathcal{A}_1$ may depend on θ. The case $m = L$, $k = L+1$, $L = 1, 2, \ldots$ corresponds to the generalized Kelvin–Voigt fluids (see [72], sect. 4.1–4.2,[73], [74]).

The derivatives involved in (2.21) equations may be both partial or substantial.

In the case of compressible continua at small displacements it is more convenient to use the Lagrangian specification and to consider the constitutive law in the form $\mathcal{L}_1(\mathbf{s}) = \mathcal{L}_2(\varepsilon(u))$ as relation for stress and strain tensors. Here $\mathcal{L}_1$ and $\mathcal{L}_2$ are linear differential operators of orders l and r respectively, $l, r = 1, 2, \ldots$. Assuming $\mathcal{L}_1(w) = w$, $\mathcal{L}_2(w) = \mathcal{B}_1 dw/dt + \mathcal{B}_2 w + \mathcal{B}_3\theta$ ($l = 0$, $r = 1$) with tensor-valued coefficients $\mathcal{B}_i$, one get typical systems of thermoviscoelasticity at small displacements (see Section 4 below).

On the other hand the appearance of L_1 terms in some models of thermoviscoelastic continua is frequently occurring. The arising in this case difficulties in L_2 theory are overcome for example with the help of the notion of renormalized solutions (Section 5).

3. FUNCTIONAL SPACES

Let $\Omega \subset R^n$ be domain with sufficiently smooth boundary $\partial\Omega$, $Q_T = [0,T] \times \Omega$. In what follows we use spaces $L_p(\Omega)$, $L_p(Q_T)$, Sobolev-Slobodetskii spaces $W_p^l(\Omega)$, Besov spaces $B_{p,r}^l(\Omega)$, spaces of Bessel potentials $H_p^\beta(\Omega)$, $p,q,r,l \in [1,+\infty)$ ([70], sect. 2.3, 4.3.1). We set $L_{pq}(Q_T) = L_p(0,T;L_q(\Omega))$, $B_{p,p}(\Omega) = B_p(\Omega)$, $W_{p,p}^{k,m}(Q_T) = W_p^{k,m}(Q_T)$, $W_{p,q}^{k,m}(Q_T)=L_p(0,T;W_q^m(\Omega)) \cap W_p^k(0,T;L_q(\Omega))$. We denote the norms in the spaces $L_2(\Omega)$, $W_2^l(\Omega)$, $L_2(Q_T)$ and $W_2^{k,m}(Q_T)$ by $|\cdot|_0$, $|\cdot|_l$, $\|\cdot\|_0$, $\|\cdot\|_{k,m}$, respectively. The notation $L_p(\Omega)^n$ (or $L_p(\Omega;R^n)$ and similar ones mean that the space consists of functions taking it values in R^n.

Let $C_0^\infty(\Omega,R^n)$ be the set of infinitely differentiable functions on Ω having compact support $\mathcal{D}'(\Omega)^n$ denote the space of distribution on $C_0^\infty(\Omega,R^n)$. By $\overset{\circ}{W}{}_p^m(\Omega)$ denote the closure of $C_0^\infty(\Omega)=C_0^\infty(\Omega,R^1)$ with respect to the norm of $W_p^m(\Omega)$ $(m>0)$ and let $W_{p,0}^m(\Omega) = W_p^m(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega)$, $m > 1/p$. Further, let $W_p^{-m}(\Omega) = (\overset{\circ}{W}{}_{p'}^m(\Omega))'$, where $m>0$, $p'=p/(p-1)$, $1<p<+\infty$, and $'$ denotes the dual space.

Let $\mathcal{V} = \{u : u \in C_0^\infty(\Omega,R^n), \operatorname{div} u = 0\}$. In what follows H and V are the closures of $\mathcal{V}$ in the norms $|\cdot|_0$ and $|\cdot|_1$ respectively. $\mathcal{P} : L_2(\Omega,R^n) \to H$ is the orthoprojector in $L_2(\Omega,R^n)$ onto the space of solenoidal functions H.

Denote by $C^r(0,T;E), r \in N$, the space of the E-valued functions which have r continuous derivatives with respect to t with usual norm. Denote by $W_p^r(0,T;E)$, $1 \leqslant p \leqslant +\infty$, $r=1,2,\ldots$, the Sobolev space of the functions for which the norm $\|u\|_{W_p^r(0,T;E)} = \sum_{m=0}^r \|u^{(m)}(t)\|_{L_p(0,T;E)}$ is finite.

4. COMPRESSIBLE CONTINUA

In this section we consider results concerning existence and uniqueness of weak solutions to a three-dimensional thermoviscoelastic system. The constitutive relations of the models are recovered by a free energy functional and a pseudopotential of dissipation. Using a fixed point argument, combined with an a priori estimates – passage to the limit technique, existence and uniqueness results for related initial-boundary values problems are established.

4.1. A system of thermoviscoelasticity of Kelvin–Voigt type (local solvability)

In this section we consider results of the paper [18] concerning local in time existence and nonlocal uniqueness of weak solutions to a three-dimensional thermoviscoelastic system.

Suppose we are given a thermoviscoelastic continuum filling a bounded domain $\Omega \subset R^3$ with smooth boundary $\partial\Omega$.

Basic functions are the displacement function $u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))$ and the temperature $\theta(t,x)$. We assume that the displacements are small and therefore identify Eulerian and Lagrangian coordinates. The medium is characterized by the following set $(\theta, \varepsilon_{ij})$ of thermodynamic parameters, where

$$\varepsilon_{ij} = \varepsilon_{ij}(u) = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i), \tag{4.1}$$

$\varepsilon(u) = (\varepsilon_{ij}(u))$ is the strain tensor.

The function of free energy $\Psi(\theta, \varepsilon(u))$ is

$$\Psi(\theta,\varepsilon(u)) = -c_s\theta\log\theta + \frac{1}{2}\varepsilon(u)K\varepsilon(u) + \alpha(\theta)\operatorname{tr}\varepsilon(u), \tag{4.2}$$

where K is the elastic tensor, $c_s > 0$ is the coefficient of heat capacity, $\alpha(\theta)$ is the thermal expansion coefficient, and $\alpha(\theta) = \alpha\theta$, $\alpha \in R^1$. Here $K\varepsilon(u)$ is defined as follows:

$$K\varepsilon(u) = \lambda \operatorname{tr}\varepsilon(u) I + 2\mu\varepsilon(u), \tag{4.3}$$

where λ and μ are the Lamé parameters.

The pseudopotential of dissipation Φ depends on $\nabla\theta$ and $\varepsilon(u_t)$ and refers to the heat flux and the evolution of deformation. It is given by

$$\Phi(\nabla\theta, \varepsilon(u_t)) = \frac{k_0}{2\theta}|\nabla\theta|^2 + \frac{1}{2}\varepsilon(u_t)B\varepsilon(u_t), \tag{4.4}$$

where $B = (b_{ij})$ is a symmetrical positively defined matrix.

The equation of motion is

$$u_{tt} - \operatorname{div}\mathbf{s} = G,\ (x,t) \in [0,T] \times \Omega. \tag{4.5}$$

Here G stands for external forces.

The equation of energy balance is

$$e_t + \operatorname{div} q = r + \mathbf{s}\cdot\varepsilon(u_t),\ (x,t) \in [0,T] \times \Omega. \tag{4.6}$$

Here r denotes external heat sources, the term $\mathbf{s}\cdot\varepsilon(u_t)$ stands for the increment of internal energy caused by mechanical factors, $\mathbf{s}$ is the stress tensor responsible for the increment of thermal energy caused by mechanical factors.

Further, the internal energy and the entropy $s = -\partial\Psi/\partial\theta$ satisfy the Helmholtz law $e = \Psi + s\theta$.

The non-dissipative part σ^{nd} of the stress tensor $\mathbf{s}$ has the form $\sigma^{nd} = \partial\Psi/\partial\varepsilon(u)$, and its dissipative part is $\sigma^{d} = \partial\Phi/\partial\varepsilon(u_t)$, so that $\mathbf{s} = \sigma^{nd} + \sigma^{d}$. The vector q is given by $q = -k_0\nabla\theta$.

Substituting the above equations into the equations of motion and energy balance, we obtain the system in Q_T:

$$c_s\theta_t - k_0\Delta\theta - \alpha\theta\operatorname{div} u_t = B\varepsilon(u_t)\varepsilon(u_t) + r, \tag{4.7}$$

$$u_{tt} - \operatorname{Div}(B\varepsilon(u_t) + K\varepsilon(u) + \alpha\theta I) = G. \tag{4.8}$$

Initial and boundary conditions are specified as follows:

$$\theta(0) = \theta_0;\ u(0) = u_0,\ u_t(0) = u_1; \tag{4.9}$$

$$k_0\partial_n\theta = h \text{ on } [0,T] \times \partial\Omega;\ u = u_t = 0 \text{ on } [0,T] \times \partial\Omega. \tag{4.10}$$

The model is justified in [39].

Equation (4.8) is based on the linear Kelvin–Voigt type law of thermoviscoelasticity (see [36], Chap. 5.4) $\mathbf{s} = B\varepsilon(u_t) + K\varepsilon(u) + \alpha\theta I$, where $\mathbf{s}$ is the stress tensor.

The above initial-boundary value problem is transformed into a related system of differential-operator equations.

Let $\mathbf{H} = L_2(\Omega)$, $\mathbf{V} = W_2^1(\Omega)$, $W = W_{20}^1(\Omega)^3$ and operators A, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{H}$ (operators of variational calculus [47], sec.2.5) be defined as

$$A\colon \mathbf{V} \to \mathbf{V}',\ \langle Au, v\rangle = \int\limits_\Omega \nabla u \nabla\, dx,\ \ u, v \in \mathbf{V};$$

$$\mathcal{A}\colon W \to W',\ \langle \mathcal{A}u, v\rangle = a(u,v),\ \ u, v \in W,$$

$$a(u,v) = \lambda \int\limits_\Omega \operatorname{div} u \operatorname{div} v\, dx + 2\mu \int\limits_\Omega \varepsilon_{ij}(u)\varepsilon_{ij}(v)\, dx;$$

$$\mathcal{B} : W \to W',\ \langle \mathcal{B}u, v\rangle = b(u,v),\ u, v \in W,\ b(u,v) = \int\limits_\Omega b_{ij}\varepsilon_{ij}(u)\varepsilon_{ij}(v)\, dx;$$

$$\mathcal{H}\colon \mathbf{H} \to W',\ \langle \mathcal{H}u, v\rangle = \int\limits_\Omega u \operatorname{div} v\, dx,\ \ u \in \mathbf{H}, v \in W.$$

Let the functions $\mathcal{R}$ and $\mathbb{G}$ be specified by

$$\langle \mathcal{R}(t), v\rangle = \int\limits_{\Omega} r(t)v\,dx + \int\limits_{\Gamma} h(t)v|_{\partial\Omega}\,dx,\ v \in V \text{ for a.a. } t \in [0,T],$$

$$\langle \mathbb{G}, v\rangle = \int\limits_{\Omega} G(t)v\,dx,\ v \in W \text{ for a.a. } t \in [0,T],$$

where

$$r \in L_2(0,T;\mathbf{H}),\ h \in L_2(0,T;L_2(\partial\Omega)),\ G \in L_2(0,T;\mathbf{H}^3), \tag{4.11}$$

and consequently,

$$\mathcal{R} \in L_2(0,T;\mathbf{V}'),\ \mathbb{G} \in L_2(0,T;\mathbf{H}^3).$$

Normalizing the constant coefficients, problem (4.7)–(4.10) is reduced to the following system of differential-operator equations

$$\theta_t + A\theta = \theta \operatorname{div} u_t + \mathcal{R} + |\varepsilon(u_t)|^2, \quad \text{in } \mathbf{V}', \tag{4.12}$$

$$u_{tt} + \mathcal{B}u_t + \mathcal{A}u + \mathcal{H}\theta = \mathbb{G}, \qquad \text{in } W' \tag{4.13}$$

with conditions (4.9)–(4.10). The solution to problem (4.12)–(4.13), (4.9)–(4.10) (Problem P_α) is seeking in the class

$$\theta \in W_2^1(0,T_0;\mathbf{V}') \cap C^0([0,T_0];\mathbf{H}) \cap L_2(0,T_0;\mathbf{V}) \equiv \Phi,$$

$$u \in W_2^2(0,T_0;\mathbf{H}^3) \cap W_\infty^1(0,T_0;W) \cap W_2^1(0,T_0;W_2^2(\Omega)^3) \equiv \Psi,$$

Theorem 1. *Let* (4.11) *and*

$$\theta_0 \in \mathbf{H},\ u_0 \in W \cap W_2^2(\Omega)^3,\ u_1 \in W \tag{4.14}$$

hold. Then, there exist $T_0 \in (0,T)$ and a unique pair of functions (θ,u) solving problem P_α in $[0,T_0]$.

The proof of Theorem 1 is reduced to the fix-point problem for the following operator $\mathcal{T}$. Take an arbitrary $u \in \Psi$ and substitute it in (4.13). Find a solution $\theta \in \Phi$ of (4.13) and substitute it in (4.12). Then find a solution $\widetilde{u} \in \Psi$ of (4.12), so that $\widetilde{u} = \mathcal{T}u$. Using the Schauder fix-point theorem for operator $\mathcal{T}$, combined with local in time contracting estimates, Theorem 1 is established.

For other results for similar systems see [40],[67], [57].

4.2. A system of thermoviscoelasticity of Kelvin–Voigt type (global solvability)

In this section we discuss results of [54]. A classical $3D$ thermoviscoelastic system of Kelvin-Voigt type is considered. The existence and uniqueness of a global regular solution is proved without small data assumption. The existence proof is based on the successive approximation method. The crucial part constitute a priori estimates on an arbitrary finite time interval, which are derived with the help of the theory of anisotropic Sobolev spaces with a mixed norm.

Let $\Omega \subset R^3$ be a bounded domain with the boundary $\partial\Omega \subset C^2$, $Q_T = \Omega \times [0,T]$. Consider the following equations:

$$u_{tt} - \operatorname{Div}[A_1\varepsilon_t + A_2(\varepsilon - \theta\alpha)] = b \text{ in } Q_T\ ; \tag{4.15}$$

$$c_v\theta\theta_t - k\Delta\theta = -\theta(A_2\alpha)\varepsilon_t + (A_1\varepsilon_t)\varepsilon_t + g,\ \text{in } Q_T, \tag{4.16}$$

where $\varepsilon \equiv \varepsilon(u)$, $\varepsilon = (\varepsilon_{ij}(u))$, $\varepsilon_{ij}(u) = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$.

Additionally,

$$u = 0,\ \mathbf{n}\cdot\nabla\theta = 0, \quad \text{on } \partial\Omega\times[0,T]; \tag{4.17}$$

$$u|_{t=0} = u_0,\ u_t|_{t=0} = u_1,\ \theta|_{t=0} = \theta_0, \quad \text{in } \Omega, \tag{4.18}$$

$\mathbf{n}$ is the outer normal for $\partial\Omega$. The vector $u\colon Q_T \to R^3$ is the displacement, $\theta\colon Q_T \to (0,\infty)$ is the absolute temperature, $\varepsilon = (\varepsilon_{ij})_{i,j=1,2,3}$ and $\varepsilon_t = ((\varepsilon_t)_{ij})_{i,j=1,2,3}$ are the linearized stress and strain tensors.

Equation (4.15) is based on the linear Kelvin–Voigt law of thermoviscoelasticity $\mathbf{s} = A_1\varepsilon_t + A_2(\varepsilon - \theta\alpha)$, where $\mathbf{s}$ is the stress tensor, A_1 and A_2 are tensors of order 4 defined by

$$\varepsilon \to A_p\varepsilon = \lambda_p \operatorname{tr}\varepsilon I + 2\mu_p\varepsilon,\ p = 1,2, \tag{4.19}$$

λ_1 and μ_1 are coefficients of viscosity, λ_2 and μ_2 are Lamé parameters, $\mu_p > 0$, $3\lambda_p + 2\mu_p > 0$, $p = 1,2$. The symmetric tensor $\alpha = (\alpha_{ij})_{i,j=1,2}$ with constant α_{ij} represents heat expansion.

The system is governed by two thermodynamic potentials: the free energy and the pseudopotential of dissipation. The free energy is specified by

$$\Psi(\varepsilon,\theta) = -\frac{1}{2}c_v\theta^2 + \frac{1}{2}\varepsilon(A_2\varepsilon) - \theta\varepsilon(A_2\alpha),\ c_v > 0,$$

then $e = \frac{1}{2}c_v\theta^2 + \frac{1}{2}\varepsilon(A_2\varepsilon)$. This gives rise to the term $c_v\theta\theta_t$ in energy equation (4.16). The presence of θ_t in energy equation (4.7) in Section 4.1 (the case of the caloric specific heat is constant c_v) in place of $\theta\theta_t$ in energy equation (4.16) (the case of the caloric specific heat is $c_v\theta$) is a serious mathematical obstacle in the proof of the global existence.

The pseudopotential of dissipation corresponding to system (4.15)–(4.18) is given by

$$\Phi(\nabla\theta, \varepsilon(u_t)) = \frac{1}{2\theta}\varepsilon_t(A_1\varepsilon_t) + \frac{k}{2}|\nabla \log\theta|^2.$$

Equation (4.16) is based on the Fourier law $q = -k\nabla\theta$, $k > 0$.

Main results.

Theorem 2. *(Existence) Let* $u_0 \in W^2_{12}(\Omega)$, $u_1 \in B^{11/6}_{12,12}(\Omega)$, $\theta_0 \in B^{5/3}_{6,6}(\Omega)$, $g \in L_\infty(0,T;L_{12}(\Omega))$, $b \in L_{12}(Q_T)$, $g \geqslant 0$, $\theta_0 \geqslant \bar\theta > 0$, θ_0 *is a constant. Then there exists a solution to problem* (4.15)–(4.18) *such that* $u \in C([0,T];W^2_{12}(\Omega))$, $u_t \in W^{2,1}_{12}(Q_T)$, $\theta \in W^{2,1}_6(Q_T)$, $\theta(t) \geqslant \theta\exp(-c_0T) \equiv \theta_* > 0$*; moreover, the following estimates are satisfied*

$$\|u\|_{C([0,T];W^2_{12}(\Omega))} \leqslant c\|u_t\|_{W^{2,1}_{12}(Q_T)};$$

$$\begin{gathered}\|u_t\|_{W^{1,2}_{12}(Q_T)} + \|\theta\|_{W^{2,1}_6(Q^T)} \leqslant \\ \leqslant \varphi\Bigg(T, \|u_0\|_{W^2_{12}(\Omega)} + \|u_1\|_{B^{11/6}_{12,12}(\Omega)} + \\ \|\theta_0\|_{B^{5/3}_{6,6}(\Omega)} + \|b\|_{L_{12}(\Omega)} + \|g\|_{L_{\infty,12}(Q_T)}\Bigg)\end{gathered} \tag{4.20}$$

where φ *is an increasing positive function of its arguments.*

Theorem 3. *(Uniqueness) Any solution to problem* (4.15)–(4.18) *satisfying* $\varepsilon(u_t) \in L_2(0,T;L_3(\Omega))$, $\theta \in L_2(0,T;L_{+\infty}(\Omega))$, $\theta_t \in L_2(0,T;L_3(\Omega))$ *is uniquely defined.*

The existence proof is based on the successive approximation method. The successive approximations are defined as follows:

$$u_{tt}^{n+1} - \mathrm{Div}\,(A_1\varepsilon(u_t^{n+1})) = A_2(\varepsilon - \theta^n\alpha)] + b; \tag{4.21}$$

$$c_v\theta^n\theta_t^{n+1} - k\Delta\theta^{n+1} = -\theta^n(A_2\alpha)\varepsilon(u_t^n) + A_1\varepsilon(u_t^n)\varepsilon(u_t^n) + g, \tag{4.22}$$

$$u^{n+1} = 0,\ \mathbf{n}\cdot\nabla\theta^{n+1} = 0, \quad \text{on } \partial\Omega\times[0,T], \tag{4.23}$$

$$u^{n+1}|_{t=0} = u_0,\ u_t^{n+1}|_{t=0} = u_1,\ \theta^{n+1}|_{t=0} = \theta_0, \text{ in } \Omega, \tag{4.24}$$

where u^n, θ^n, $n = 0, 1, \cdots$ are treated as given, $u^0|_{t=0} = u_0,\ u_t^0|_{t=0} = u_1,\ \theta^0|_{t=0} = \theta_0$.

It is established that the initial-value problems for (4.21) and (4.22) have solutions u^n and θ^n, and the sequence u^n, θ^n converges to the solution u, θ of (4.15)–(4.18) on a small interval $0 \leqslant t \leqslant T_0$. The apriori estimates allow to extend the solvability on the whole $[0, T]$.

The crucial part constitute a priori estimates on an arbitrary finite time interval, which are derived with the help of the theory of anisotropic Sobolev spaces with a mixed norm. The proof heavily relies on properties of solutions of parabolic system (4.21) and parabolic equation (4.22) in anisotropic Sobolev spaces $W^{2,1}_{p,p_0}(Q_T)$, $p, p_0 \in [1, \infty)$ (see [46], [31]).

It is worth to remark that (u, θ) could be found as a fixed point of a generated by approximation process operator (similar to $\mathcal{T}$ in previous section.)

Problem (4.15)–(4.18) was also considered in [60], [12] under different assumptions about A_k.

4.3. Remarks

Let us mention other papers dealing with multiphysics coupled system where, beside the thermomechanical part, also the phase-field, electro-magnetic and diffusive processes are involved. In this papers other approaches except the fixed-point arguments are used.

In [19] a thermo-mechanical nonlinear model describing hydrogen storage by use of metal hydrides is proposed. The model leads to a phase transition problem in terms of three state variables: the temperature, the phase parameter representing the fraction of one solid phase, and the pressure. The main difficulty in investigating the resulting system relies on the presence of the squared time derivative of the order parameter in the energy balance equation. The global existence of a weak solution to the problem is proved by exploiting sharp estimates on parabolic equations with right hand-side in L_1. Some results on stability and steady state solutions are also given. The proof of the results based on a regularization of the problem, time-discretization of the regularizied problem, application of direct method of the Calculus of Variations for its solvability and the subsequent passage to the limit using weak and weak star compactness arguments.

The case of for a Kelvin-Voigt-type systems with heat capacity dependent both on temperature and on the strain is considered in [62]. The existence of a weak solution is proved by means of a suitably regularized Rothe method and a subsequent limit passage.

In [63] the existence of weak solutions of the system of thermomechanics of hydrogen storage in metallic hydrides is proved. A semi-implicit discretisation in time scheme which decouples the system to a suitable sequence of convex minimization problems combined with a diffusion equation provides a rather efficient conceptual numerical strategy.

5. INCOMPRESSIBLE CONTINUA

The traditional mathematical model for the description of heat gravitational convection in viscous liquids is the system of Oberbeck-Boussinesq equations ([41], sect. IIIV.54), which is a result of reduction of the complete equations of mechanics of continuous media under the following conditions: a density of liquid linearly depends only on its temperature (liquid is isothermal incompressible); a velocity field is considered to be solenoidal, however, in the equation of momentum approximately takes into account the change of a density; the contribution of dissipation and forces of pressure in the description of heat flow is negligible; a dynamic viscosity ratio, coefficients of thermal conductivity and specific heat capacity are assumed to be constant.

Different assumptions about the nature of the dependence of the coefficients on the velocity, temperature and additional parameters in specific physical problems lead to a variety of Boussinesq's type models.

Below we use lower-case letters for velocity and temperature (in spite of the Eulerian specification).

5.1. Boussinesq's type equations with nonlinear viscosity

This section deals with results of [8].

Let $\Omega \subset R^N$, $(N = 2, 3)$ be a bounded Lipschitz domain, $Q_T = \Omega \times [0, T]$, $0 < T < +\infty$. Consider the problem

$$\partial v/\partial t + v_i \partial v/\partial x_i - 2\mathrm{Div}(\mu(\theta)D(v)) + \nabla p = F(\theta), \text{ in } Q_T, \tag{5.1}$$

$$\partial(b(\theta))/\partial t + v_i \partial(b(\theta))/\partial x_i - \Delta\theta = 2\mu(\theta)|D(v)|^2, \text{ in } Q_T, \tag{5.2}$$

$$\mathrm{div}\, v = 0, \text{ in } Q_T, \tag{5.3}$$

$$v = 0, \theta = 0, \text{ on } \partial\Omega \times [0, T], \tag{5.4}$$

$$v|_{t=0} = v_0, b(\theta)|_{t=0} = b(\theta_0), \text{ in } \Omega, \tag{5.5}$$

Velocity $v\colon Q_T \to R^N$ and temperature $\theta\colon Q_T \to R^1$ are the unknowns. Further, $D(v) = (D_{ij}(v))$, $D_{ij}(v) = D_{ij} = \frac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$ is the strain velocity tensor, $|D(v)|^2 = D_{ij}D_{ij}$, the constitutive law is $\sigma = -pI + 2\mu(\theta)D(v)$, p is the pressure, the function μ is positive, bounded, and continuous on R^1, $\mu_0 \in L_2(\Omega)^N$, $v_0 \cdot \mathbf{n} = 0$ on $\partial\Omega$, $F\colon R^1 \to R^N$ is the buoyancy force depending on the temperature, function b, characterizing dependance of the internal energy on the temperature, is an *strictly* increasing C^1-function, $b(0) = 0$, $b'(r) \geqslant \alpha_1\, \forall r \in R^1$, $\alpha_1 > 0$, $b(\theta_0) \in L_1(\Omega)$.

Equation (5.1) is the conservation equation of momentum. Equation (5.2) is the energy conservation equation, in which the right hand side $\mu(\theta)|D(u)|2$ is the dissipation energy.

Specify the differences between (5.1)–(5.5) and Oberbeck-Boussinesq system:

- the viscosity coefficient μ and the external forcing term F are temperature-dependent (with nonlinear dependence);

- the internal energy is also assumed to be nonlinear with respect to the temperature and this affects the time derivative term in the temperature equation;

- there is a right hand side in the energy conservation equation which is quadratic in the spatial gradient of the velocity field.

One of the difficulties arising in the study of the problem (5.1)–(5.5) in a weak setting is due to the fact that the right-hand side of (5.2) belongs to $L_1(Q)$.

The notion of weak-renormalized solution proved useful in the study of elliptic and parabolic problems with data belonging to L_1. This notion has been introduced by R.-J. DiPerna and P.-L. Lions in [34] and [35] for the study of Boltzmann equations. This notion was then adapted to various types of elliptic and parabolic problems with L_1 data (see [48]). See [13], [13], [14], [29], [30], [16], [17], [33], [56] for application of this notion to various types of elliptic and parabolic problems.

The investigation of the problem (5.1)–(5.5) is carried out in the framework of theory of weak-renormalized solutions.

Define the truncation function as $T_k(s) = s$ if $|s| \leqslant k$, and $T_k(s) = k\,\mathrm{sign}(s)$ if $|s| > k$. Let

$L_{p,\sigma}(\Omega)$ be the closure of $\mathcal{V}$ in $L_p(\Omega)^N$, $p \geqslant 1$,

$$a_\theta(u,v) = \frac{1}{2}\int\limits_\Omega \mu(\theta)(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})\frac{\partial v_j}{\partial x_i}dx,\ \forall\, u,v \in V,$$
$$d(u,v,w) = \int\limits_\Omega u_j \frac{\partial v_i}{\partial x_j} w_j dx \quad \forall\, u,v \in V,\ \forall\, w \in V \cap L_{N,\sigma}(\Omega).$$

Definition 5.1. *The* weak-renormalized *solution of* (5.1)–(5.5) *is a pair* (v,θ) *such that:*

$$v \in L_2(0,T;V) \cap L_\infty(0,T;H);$$
$$T_k(\theta) \in L_2(0,T;\overset{\circ}{W}{}^1_2(\Omega))\,\forall k \geqslant 0 \text{ and } b(\theta) \in L_\infty(0,T;L_1(\Omega));$$
$$\int\limits_{(x,t)\in Q_T\,:\,n\leqslant|b(\theta)(x,t)|\leqslant n+1} b'(\theta)|\nabla\theta|^2 dxdt \to 0 \text{ as } n \to \infty;$$
$$\langle v_t, w\rangle_{L_{2,\sigma}(\Omega)} + a_0(v,w) + d(v,v,w) = \langle F(\theta), w\rangle\,\forall w \in V \cap L_{N,\sigma}(\Omega),$$
$$v(t=0) = v_0 \text{ a.e. in } \Omega;$$

$\forall\, S \in C^\infty(R^1)$ *with compact support*

$$\partial S(b(\theta))/\partial t + \operatorname{div}(vS(b(\theta))) - \operatorname{div}(S'(b(\theta))\nabla\theta) + S''(b(\theta))b'(\theta)|\nabla\theta|^2 = 2\mu(\theta)|D(v)|^2 S'(b(\theta)) \text{ in } \mathcal{D}'(Q_T), \tag{5.6}$$
$$S(b(\theta))(t=0) = S(b(\theta_0)) \text{ in } \Omega. \tag{5.7}$$

Here v satisfies (5.1) in the weak sense. As to the notion of renormalized solution, let as remark that equation (5.6) is formally obtained through pointwise multiplication of equation (5.2) by $S'(b(\theta))$. In other terms, equation (5.6) is nothing but

$$-\int\limits_{Q_T} S(b(\theta))\varphi_t\,dxdt + \int\limits_{Q_T} v_i\partial S(\theta)/\partial x_i\varphi\,dxdt + \int\limits_{Q_T} S''(b(\theta))b'(\theta)|\nabla\theta|^2\varphi\,dxdt +$$
$$\int\limits_{Q_T} S'(b(\theta))\partial\theta/\partial x_i\partial\varphi/\partial x_i\,dxdt = 2\int\limits_{Q_T} \mu(\theta)|D(v)|^2 S'(b(\theta))\varphi\,dx\,dt.$$

for any $S \in W^2_\infty(R^1)$ such that $\operatorname{supp}(S')$ is compact and for any $\varphi \in C_0^\infty((0,T)\times\Omega)$ such that $S'(\theta)\varphi \in L_2(0,T;\overset{\circ}{W}{}^1_2(\Omega))$. This formally corresponds to using the test function $\varphi S(b(\theta))$ in (5.2), but this is only formal.

The main result of [8] runs as follows:

Theorem 4. *Let* $N = 2$. *Assume that*

$$|F(r)| \leqslant a + b|r|^\alpha,\ \alpha \geqslant 0,\ \mu \geqslant 0,\ 0 \leqslant 2\alpha \leqslant 3,\ v_0 \in H,$$

θ_0 *is measurable on* Ω *and* $b(\theta_0) \in L_1(\Omega)$. *Then:*

- *if* $0 \leqslant 2\alpha \leqslant 1$, *there exists at least a weak-renormalized solution of problem* (5.1)–(5.5).
- *if* $1 \leqslant 2\alpha \leqslant 3$, *there exists a real positive number* η *such that if* $a + \|u_0\|_{L_2(\Omega)^2} + \|b(\theta_0)\|_{L_1(\Omega)} \leqslant \eta$, *there exists at least a weak-renormalized solution of problem* (5.1)–(5.5).

Theorem 5. *Let* $N = 3$. *Assume that* F *is continuous and bounded,* $u_0 \in V$, θ_0 *satisfies conditions of Theorem 4. Then there exists at least a weak-renormalized solution of the system* (5.1)–(5.5) *provided*

$$\|v_0\|_{W^1_2(\Omega)^3} + \|F\|_{L_\infty(R)^3} \leqslant \eta$$

for sufficiently small positive number η.

The proof of the main result involves consideration of the ε-approximate problem (5.1)–(5.5), in which b, F are replaced by b_ε, F_ε. Here b_ε is C^2-approximations of b, F_ε is a smooth bounded approximations of F. Take an arbitrary θ from suitable $L = L_r(0,T;L_q(\Omega))$, $r,q \in [1,+\infty)$ and substitute it in ε-approximate (5.1). Find its solution u. Then find a weak-renormalized solution $\widetilde{\theta}$ of ε-approximate (5.2), so that $\widetilde{\theta} = \psi_\varepsilon(\theta)$. Using the Schauder fix-point theorem for operator ψ_ε one finds a weak-renormalized solution $(\theta_\varepsilon, u_\varepsilon)$ to ε-approximate problem (5.1)–(5.5).

The existence of a weak-renormalized solution of the coupled system (5.1)–(5.5) is then obtained by passing to the limit in this approximate problem.

Other results for similar models can be found in [28], [29], [33], [37], [42], [51], [52], [48].

5.2. Boussinesq's type equations with constant viscocity

Here we consider the results of [11].

Let $\Omega \subset R^2$ be a bounded domain, $\partial\Omega \in C^2$. Consider the nonlinear equations of Boussinesq's type:

$$\partial v/\partial t + v_i \partial v/\partial x_i - \Delta v + \nabla p = F(\theta);\ \operatorname{div} v = 0,\ \text{in } Q_T; \tag{5.8}$$

$$\partial\theta/\partial t - \Delta\theta + v_i \partial\theta/\partial x_i = \delta |D(v)|^2 + f,\ \text{in } Q_T; \tag{5.9}$$

$$v(t=0) = v_0,\ \theta(t=0) = \theta_0 \text{ in } \Omega;\ v = 0,\ \theta = 0,\ \text{in } \partial\Omega \times [0,T]. \tag{5.10}$$

The system (5.8)–(5.10) is a special case of (5.1)–(5.5) for linear b, $\delta = 1$ and $\mu \equiv 1/2$ that allows to get uniqueness of weak-renormalized solutions additionally to the existence.

Theorem 6. *Suppose that $\delta = 0$. Let $\theta_0 \in L_1(\Omega)$, $v_0 \in V$, $f \in L_1(Q_T)$, $F(r)$ be continuous, $|F(r)| \leqslant a + b|r|^\alpha$, $r \in R$, $a, b > 0$ and the Lipschitz condition*

$$|F(r) - F(s)| \leqslant L|r - s|(1 + |r|^\beta + |s|^\beta),\ r, s \in R^1,\ 0 \leqslant \beta < 1, \tag{5.11}$$

hold true. Then, if $\|\theta\|_{L_1(\Omega)}$, $\|f\|_{L_1(Q_T)}$ are sufficiently small, there exists a unique weak-renormalized solution to problem (5.8)–(5.10).

Here the existence of a weak-renormalized solution of the problem (5.8)–(5.10) is insured by Theorem 4.

Consider the case $\delta = 1$. Theorem 4 gives the existence of at least a weak-renormalized solution to (5.8)–(5.10) such that $F(\theta) \in L_2(Q_T)$. Using the smooth character of data, it is shown that this solution is actually a weak solution (which means that θ is a weak solution of the heat equation). Then uniqueness of a small solution is established provided the Lipschitz condition on F holds true.

Theorem 7. *Suppose that $\delta = 1$. Let $\theta_0 \in L_1(\Omega)$, $u_0 \in V \cap W_2^2(\Omega)^N$, $f \in L_2(Q_T)$, $F(r)$ be continuous and $|F(r)| \leqslant a + b|r|^\alpha$, $r \in R$, $a, b > 0$. Further, suppose that either $0 \leqslant 2\alpha < 1$ or*

$$1 < 2\alpha < 3 \text{ and } a,\ \|v_0\|_V,\ \|f\|_{L_2(Q_T)} \text{ are sufficiently small.}$$

Then problem (5.8)–(5.10) *admits at least a weak solution.*

Moreover, if (5.11) *holds true, there is uniqueness of the weak solution of the problem* (5.8)–(5.10) *for sufficiently small solutions. More precisely, there exists $R > 0$ such that if (v^1, θ_1) and (v^2, θ_2) are two weak solutions of* (5.8)–(5.10) *satisfying $\|\theta_i\|_{L_4(Q_T)} \leqslant R$, $\|\partial\theta_i/\partial x_j\|_{L_2(Q_T)} \leqslant R$, $\|\partial v^i/\partial x_j\|_{L_4(Q_T)} \leqslant R$, $i, j = 1, 2$, then $\theta_1 = \theta_2$, $v^1 = v^2$.*

Other results for similar models can be found in [29], [10].

5.3. Boussinesq's type equations with nonlinear thermal diffusion

In this section we consider the results of [32]. There are some fluids such as lubrificants or some plasma flow for which the energy conservation equation is nonlinear parabolic equation. Below the existence and uniqueness of weak solutions for this kind of models are given.

Let $Q_T = [0,T] \times \Omega$ be a bounded and connected domain, where $\Omega \subset R^N$, $N = 2,3$ and $\partial\Omega \in C^1$. Consider the equations

$$v_t + v_i \partial v/\partial x_i - \mathrm{Div}(\mu(\theta)D(u)) + \nabla p = F(\theta); \ \mathrm{div}\, v = 0; \tag{5.12}$$

$$\theta_t + v_i \partial\theta/\partial x_i - \Delta\varphi(\theta) = 0, \tag{5.13}$$

where v is the velocity, θ is the temperature, p is the pressure, $F(\theta)$ is the buoyancy force, and $\varphi(\theta) = \varphi_D$ on $\partial\Omega \times [0,T]$, $v(\cdot,0) = v_0$ and $\theta(\cdot,0) = \theta_0$ in Ω.

Note that the equation in (5.13) is the energy conservation equation transformed by means of a suitable change of temperature function. The function φ takes into account interaction between non-constant specific heat and heat conductivity (slow and fast diffusion). The model is justified in [49].

It is assumed that:

$$\varphi \in C([0,\infty)) \cap C^1((0,\infty)),\ \varphi(0) = 0,\ \varphi \text{ is nondecreasing};$$
$$F \in C^{0,1}_{\mathrm{loc}}([0,\infty); R^N);$$
$$\mu \in C^{0,1}_{\mathrm{loc}}([0,\infty)),\ m_0 \leqslant \mu(s) \leqslant m_1,\ 0 < m_0 \leqslant m_1;$$
$$u_0 \in H,\ \theta_0 \in L_\infty(\Omega),\ \theta_0 \geqslant 0;$$
$$\varphi_D \in L^2(0,T; \overset{\circ}{W}{}^1_2(\Omega)) \cap W^1_2(0,T; L_2(\Omega)) \cap L_\infty(Q_T).$$

If $\mu' \neq 0$ or $F' \neq 0$, then φ^{-1} is supposed to satisfy the Hölder condition with some $\alpha \in (0,1)$.

Let

$$a_\theta(u,v) = \frac{1}{2}\int\limits_\Omega \mu(\theta)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial v_j}{\partial x_i}dx,\ u,v \in V \text{ in } \theta \in L_\infty(Q_T);$$
$$b(u,v,w) = \int\limits_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx \text{ for } u,v \in V,\ w \in V \cap L_{N,\sigma}(\Omega).$$

Definition 5.2. *The* weak solution *of* (5.12)-(5.13) *is a pair* (v,θ) *such that*

1. $v \in L_2(0,T;V) \cap L_\infty(0,T;H)$, $\theta \in L_\infty(Q_T)$, $\theta_t \in L_2(0,T;\overset{\circ}{W}{}^1_2(\Omega))$, $\varphi(\theta) \in \varphi_D + L_2(0,T;\overset{\circ}{W}{}^1_2(\Omega))$;
2. $v(\cdot,0) = v_0$ *a.e. in* Ω, *and for a.e.* $t \in [0,T]$ *the identity holds:*

$$\langle v_t, w\rangle_H + a_\theta(v,w) + b(v,v,w) = \langle F(\theta), w\rangle_H \ \forall\, w \in V \cap L_{N,\sigma}(\Omega);$$

3. *for* $\forall\, \xi \in L_2(0,T;\overset{\circ}{W}{}^1_2(\Omega))$ *and for* $\forall\, \psi \in L_2(0,T;\overset{\circ}{W}{}^1_2(\Omega)) \cap W^1_1(0,T;L_2(\Omega))$ *with* $\psi(T) = 0$

$$\int\limits_0^T \langle\theta_t, \xi\rangle_{V',V}\, dt + \int\limits_0^T\int\limits_\Omega (\nabla\varphi(\theta) - \theta v)\nabla\xi\ dtdx = 0;$$

$$\int\limits_0^T \langle\theta_t, \psi\rangle_{W_2^{-1}(\Omega),\overset{\circ}{W}{}^1_2(\Omega)}\, dt + \int\limits_0^T\int\limits_\Omega (\theta - \theta_0)\psi_t\, dtdx = 0.$$

The main result of [32] is the proof of existence of a weak solution of (5.12)-(5.13) such that

Theorem 8. *Ther exists a weak solution of* (5.12)-(5.13) *such that*

$$v \in C([0,T];V'),\ \theta \in C([0,T];W_2^{-1}(\Omega)).$$

Moreover, if there exist non-negative constants k, m, λ_1, λ_2 such that

$$k \geqslant \theta_0 \geqslant m \geqslant 0, \quad \varphi(k\exp(\lambda_0 t)) \geqslant \psi_D(\cdot,t) \geqslant \varphi(m\exp(-\lambda_1 t)) \geqslant 0,$$

then

$$k\exp(\lambda t) \geqslant \theta(\cdot,t) \geqslant m\exp(-\lambda t) \geqslant 0 \text{ for } t \in [0,T] \text{ and a.e. in } \Omega.$$

In the case $N=2$, $\mu \equiv 1$, $\varphi^{-1} \in C^{0,1}([0,\infty))$ there exists the unique weak solution of (5.12)-(5.13).

The existence proof is based on the successive approximation method similar to one in section 4.2. The solvability of approximative equation (5.12) relies in Galerkin's method. The solvability of approximative equation (5.13) uses results in [3] on solvability nonlinear parabolic problems. Passage to the limit in the successive approximations gives a weak solution of the coupled system.

5.4. Non-isothermal solidification problem with melt convection

The Allen-Cahn and Cahn-Hilliard equations ([2], [24]) have been widely used in many complicated moving interface problems in two-phase flow of viscous fluids and alloys through a phase-field approach.

This section deals with results of [38] on a phase-field model for solidification processes in a molten material which is assumed to behave as an incompressible fluid with variable viscosity.

Let $\Omega \subset R^N$, $N = 2,3$ be a bounded open domain, $\partial\Omega \subset C^2$.

Consider in $Q_T = \Omega \times [0,T]$ the problem

$$\varphi_t + v_i \partial\varphi/\partial x_i - \xi^2 \Delta\varphi = \varphi(\varphi-1)(1-2\varphi) + \theta; \tag{5.14}$$

$$\theta_t + v_i \partial\theta/\partial x_i - \operatorname{div}(k(\varphi,\theta)\nabla\theta) = \nu(\varphi,\theta)|D(v)|^2, \text{ in } Q_T; \tag{5.15}$$

$$v_t + v_i \partial v/\partial x_i - \operatorname{Div}(\nu(\varphi,\theta)D(v)) + \nabla p = f; \operatorname{div} v = 0; \tag{5.16}$$

$$\varphi = 0,\ \theta = 0,\ v = 0, \qquad \text{on } \partial\Omega \times [0,T]; \tag{5.17}$$

$$\varphi(x,0) = \varphi_0(x),\ \theta(x,0) = \theta_0(x),\ v(x,0) = v_0(x), \text{ in } \Omega. \tag{5.18}$$

Model (5.14)–(5.18) is justified in [5], [9], and [22]. The unknowns are the phase-field function φ, the temperature θ, the velocity field u and the hydrostatic pressure p; ξ is a positive constant related to the width of the transitions layers; k and ν are strictly positive functions that depend on φ and θ and must be viewed as a heat diffusion and a kinematic fluid viscosity, respectively; f is an external field; φ_0, θ_0 and u_0 are given functions.

The structure of this system is typical in non-isothermal solidification problems with melt convection. This model couples the Boussinesq system to the Allen-Cahn equation for a non-mechanical phase-field variable.

Notice that due to the nonlinear right-hand side, that only belongs to $L_1(Q)$, the problem (5.14)–(5.18) is considered in the framework of theory of weak-renormalized solutions. For this reason, the notion of renormalized solutions adapted to the problem (5.14)–(5.18) is given below.

Introduce the following notations. Put $L(0,T;\Omega) = \{v \in L_\infty(0,T;L_1(\Omega)) : T_R(v) \in L_2(0,T;H_0^1(\Omega))\ \forall R > 0,\ \lim\limits_{n\to\infty} \frac{1}{n} \int\limits_{A_n(v)} |\nabla v|^2 dxdt = 0\}$, $A_n(v) = \{(x,t) \in Q : n \leqslant |v(x,t)| \leqslant 2n\}$.

Definition 5.3. *The triple* (φ, θ, v) *is called a* weak-renormalized *solution of problem* (5.14)–(5.18) *if*

1. $v \in L_\infty(0,T;H) \cap L_2(0,T;V)$, $\varphi \in L_\infty(0,T;L_2(\Omega)) \cap L_2(0,T;H_0^1(\Omega)) \cap L_4(Q)$ *and* $\theta \in L(0,T;\Omega)$*;*

2. φ *is a weak solution of* (5.14) *in usual sense and* $\varphi|_{t=0} = 0$*;*

3. v *is a weak solution of* (5.16) *in usual sense (together with* $p \in \mathcal{D}'(\Omega)$*) and* $v|_{t=0} = 0$*;*

4. *for any* $\beta \in W_\infty^2(R)$ *such that* $\operatorname{supp}\beta$ *is a compact set and for any* $\eta \in C^1([0,T];H_0^1(\Omega)) \cap L_\infty(Q)$ *such that* $\eta|_{t=T} = 0$ *the following condition holds:*

$$
-\iint\limits_Q \beta(\theta)h_t\,dxdt + \iint\limits_Q k(\varphi,\theta)\nabla\beta(\theta)\nabla\eta\,dxdt +
+\iint\limits_Q k(\varphi,\theta)(\nabla\theta\ \nabla\beta'(\theta))\eta\,dxdt - \iint\limits_Q ((v\cdot\nabla)\beta'(\theta))\eta dxdt =
= \iint\limits_Q \beta'(\theta)\nu(\varphi,\theta)|D(v)|^2\eta\,dxdt + \int\limits_\Omega \beta(\theta_0)\eta(x,0)\,dx.
$$

The main result of [38] runs as follows.

Theorem 9. *Suppose that* $N = 2$, $f \in L_2(Q)^2$, $\varphi_0 \in L_2(\Omega)$, $v_0 \in H$, $\theta_0 \in L_1(\Omega)$, $\nu, k \in C_0(R\times R)$, $0 < \nu_1 < \nu \leqslant \nu_2$, $0 < k_1 \leqslant k \leqslant k_2$*; then there exists at least one weak-renormalized solution of problem* (5.14)–(5.18).

The proof relies on the solvability of the regularized problem $(5.14)_\varepsilon$–$(5.18)_\varepsilon$, i.e. the problem (5.14)–(5.18) in which the right-hand side in (5.15) and θ_0 are changed by $g_\varepsilon = T_{1/\varepsilon}(\nu(\varphi_\varepsilon,\theta_\varepsilon)|D(v_\varepsilon)|^2)$ and $\theta_{0\varepsilon} = T_{1/\varepsilon}(\theta_0)$, $\varepsilon > 0$, respectively.

Consider the mapping $\Lambda_\varepsilon : L_1(Q)^2 \to L_1(Q)^2$ that associates to each (φ,θ), first, the unique solution v_ε to (5.16); then, the unique solution θ_ε to $(5.15)_\varepsilon$, finally, the unique solution φ_ε to (5.16) for $v = v_\varepsilon$ and $\theta = \theta_\varepsilon$, so that $(\varphi_\varepsilon,\theta_\varepsilon) = \Lambda_\varepsilon(\varphi,\theta)$.

The solvability of the defining $(5.14)_\varepsilon$–$(5.18)_\varepsilon$ problems is obtained from a standard Galerkin scheme, the existence of solution to $(5.14)_\varepsilon$–$(5.18)_\varepsilon$ follows from Schauder's fixed-point theorem for operator Λ_ε.

The solvability of (5.14)–(5.18) is established by virtue of passage to the limit by $\varepsilon = 1/n$, $n \to +\infty$.

The problems of solvability for $N = 3$ are discussed.

Other results for similar problems can be found in [23] and [50].

5.5. Some remarks

Let us mention some papers dealing with Navier-Stokes-Fourier-type equations which characterizes the Newtonian fluids under heat-conducting effects.

In [20] the large-data and long-time existence of a suitable weak solution to an initial and boundary value problem driven by a system consisting of the Navier-Stokes equations with the viscosity polynomially increasing with a turbulent kinetic energy that evolves according to an evolutionary convection diffusion equation (Navier-Stokes-Fourier problem) with the L_1-summable right-hand side are established. The L_1 difficulties are overcome by means of a specific notion of a weak solution, the application of a Galerkin type method for the regularized problem and with the subsequent passage to the limit.

In [21] the mathematical properties of unsteady 3D internal flows of incompressible fluids are investigated. The model is expressed through a system of equations representing the balance of mass, the balance of linear momentum, the balance of energy and the equation for the entropy production. The viscosity and the thermal conductivity are assumed to be dependant on the pressure and the temperature. Supposing Navier's slip condition at the impermeable boundary the long-time existence of a (suitable) weak solution by the large data are established.

The paper [27] deals with the existence of strong 2D solutions to the Navier-Stokes-Fourier system. The nonstationary Navier-Stokes system for an incompressible homogeneous fluid with temperature dependent viscosity is completed by the equation of balance of energy which includes the term of dissipative heating. The regularity of solutions to the problems under study is proved through compactness methods and fixed point arguments, instead assuming the existence of weak solutions to the problems. The proof relies on sufficient smallness of derivatives of viscosity and heat-conductivity coefficients and sufficient smoothness of the data.

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