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ON THE DESCRIPTION OF SOME ANISOTROPIC VISCOUS FLUIDS BY THE METHODS OF STOCHASTIC ANALYSIS ON THE GROUPS OF DIFFEOMORPHISMS

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Abstract: in the framework of Lagrangian approach to hydrodynamics we suggest a special stochastic perturbation of the flow of perfect incompressible fluid on flat n-dimensional torus $\mathcal{T}^n$ and obtain the description of viscous incompressible fluid with viscous term in the form of some second order differential operator more general than Laplacian. This model describes anisotropic fluids. We show that transition to Euler description of such a fluid yields the solution of an analogue of Navier-Stokes equation without external force.

Key words and phrases: group of diffeomorphisms; flat torus; stochastic perturbation; perfect incompressible fluid; Reynolds equation; Navier-Stokes equation.

INTRODUCTION

In this paper we present a development of idea, suggested in [1] (see also [2]), of the use of special stochastic perturbations of flows of perfect fluids to obtain stochastic flows whose expectation describes the motion of viscous fluids. This approach is based on machinery of mean derivatives (see [3], [4], [5] and on geometry of groups of Sobolev diffeomorphisms (see [6]). The flow of perfect fluid is considered as a curve in the group of diffeomorphisms and its stochastic perturbation satisfies a special equation in terms of mean derivatives.

In [1] this idea is realized for classical viscous fluids for which Euler's description is given by the Navier-Stokes equation. Here we deal with the fluids for which in Euler's description the Laplacian is replaced by a more general second order differential operator. We interpret such equations as the ones describing anisotropic fluids.

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1. PRELIMINARIES

Consider a stochastic process $\xi(t)$ in $\mathbb{R}^n$, where $t \in [0,T]$, given on a certain probability space $(\Omega, \mathcal{F}, P)$, and such that $\xi(t)$ is an L^1-random variable for all t. The "present" for $\xi(t)$ is the least complete σ-subalgebra $\mathcal{N}_t^{\xi}$ of $\mathcal{F}$ that includes preimages of the Borel set of $\mathbb{R}^n$ under the map $\xi(t) : \Omega \to \mathbb{R}^n$. The least complete σ-subalgebra that includes preimages of the Borel set of $\mathbb{R}^n$ under the map $\xi(t) : \Omega \to \mathbb{R}^n$ for $s \leqslant t$ is called the "past"σ-algebra and is denoted by $\mathcal{P}_t^{\xi}$. The least complete σ-subalgebra that includes preimages of the Borel set of $\mathbb{R}^n$ under the map $\xi(t) : \Omega \to \mathbb{R}^n$ for $s \geqslant t$ is called the "future"σ-algebra and is denoted by $\mathcal{F}_t^{\xi}$.

We denote by E_t^{ξ} the conditional expectation with the respect to $\mathcal{N}_t^{\xi}$.

Below we most often deal with the diffusion processes of the form

$$\xi(t) = \xi_0 + \int_0^t a(s, \xi(s))ds + \mathbf{B}w(t) \tag{1}$$

in $\mathbb{R}^n$ and flat torus $\mathcal{T}^n$ as well as natural analogue of such processes on groups of diffeomorphisms. In (1) $w(t)$ is a Wiener process adapted to $\xi(t)$; $a(t,x)$ is a vector field; $\mathbf{B}$ is a constant linear operator in $\mathbb{R}^n$.

Following [3], [4], [5] and [1] we give the following definitions:

Definition 1. *The forward mean derivative $D\xi(t)$ of the process $\xi(t)$ at t is the L^1-random variable of the form*

$$D\xi(t) = \lim_{\Delta t \to 0+} E_t^\xi \left(\frac{\xi(t+\Delta t) - \xi(t)}{\Delta t} \right) \tag{2}$$

where the limit is supposed to exist in $L^1(\Omega, \mathcal{F}, P)$ and $t \to 0+$ means that $t \to 0$ and $\Delta t > 0$

Definition 2. *The backward mean derivative $D_*\xi(t)$ of the process $\xi(t)$ at t is the L^1-random variable of the form*

$$D_*\xi(t) = \lim_{\Delta t \to 0+} E_t^\xi \left(\frac{\xi(t) - \xi(t-\Delta t)}{\Delta t} \right) \tag{3}$$

where the limit is supposed to exist in $L^1(\Omega, \mathcal{F}, P)$ and $t \to 0+$ means that $t \to 0$ and $\Delta t > 0$

From the properties of conditional expectation it follows that $D\xi(t)$ and $D_*\xi(t)$ can be represented as compositions of $\xi(t)$ and Borel measurable vector fields (called regressions)

$$Y^0(t,x) = \lim_{\Delta t \to 0+} E_t^\xi \left(\frac{\xi(t+\Delta t) - \xi(t)}{\Delta t} | \xi(t) = x \right) \tag{4}$$

$$Y_*^0(t,x) = \lim_{\Delta t \to 0+} E_t^\xi \left(\frac{\xi(t) - \xi(t-\Delta t)}{\Delta t} | \xi(t) = x \right) \tag{5}$$

on $\mathbb{R}^n$. This means that $D\xi(t) = Y^0(t, \xi(t))$ and $D_*\xi(t) = Y_*^0(t, \xi(t))$. We notice that for a process of type (1), $D\xi(t) = a(t, \xi(t))$ and so $Y^0(t,x) = a(t,x)$.

Let $Z(t,x)$ be a $\mathcal{C}^2$-smooth vector field on $\mathbb{R}^n$.

Definition 3. *The L^1-limits of the form*

$$DZ(t, \xi(t)) = \lim_{\Delta t \to 0+} E_t^\xi \left(\frac{Z(t+\Delta t, \xi(t+\Delta t)) - Z(t, \xi(t))}{\Delta t} \right) \tag{6}$$

$$D_*Z(t, \xi(t)) = \lim_{\Delta t \to 0+} E_t^\xi \left(\frac{Z(t, \xi(t)) - Z(t-\Delta t, \xi(t-\Delta t))}{\Delta t} \right) \tag{7}$$

are called forward and backward, respectively, mean derivatives of Z along $\xi(t)$ at time instant t.

Certainly $DZ(t, \xi(t))$ and $D_*Z(t, \xi(t))$ can be represented in terms of corresponding regressions, analogously to (4), (5). We denote these regressions by DZ and D_*Z (see e.g., [1]).

Lemma 1. *For the process (1) in $\mathbb{R}^n$ the following formulae take place:*

$$DZ = \frac{\partial}{\partial t} Z + (Y^0 \cdot \nabla) Z + \widetilde{\mathfrak{B}} Z \tag{8}$$

$$D_*Z = \frac{\partial}{\partial t} Z + (Y_*^0 \cdot \nabla) Z - \widetilde{\mathfrak{B}} Z \tag{9}$$

where $\nabla = (\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n})$ and $\widetilde{\mathfrak{B}} = \frac{1}{2} \widetilde{\mathfrak{B}}^{ij} \frac{\partial^2}{\partial x^i \partial x^j}$ is the second order differential operator with the matrix $(\widetilde{\mathfrak{B}}^{ij}) = \mathbf{B}\mathbf{B}^*$.

2. THE MAIN IDEA

The main idea of description of viscous hydrodynamics in the language of mean derivatives is as follows (see, e.g., [1], [2]). We deal with the fluids moving on a flat n-dimensional torus $\mathcal{T}^n$. It is the quotient space of $\mathbb{R}^n$ with the respect the integral lattice where the Riemannian metric is inherited from $\mathbb{R}^n$. Consider the vector space $Vect^{(s)}$ of all H^s-vector fields ($s > n/2 + 1$) on $\mathcal{T}^n$. Introduce the L^2-inner product in $Vect^{(s)}$ by the formula

$$(X, Y) = \int_{\mathcal{T}^n} \langle X(m), Y(m)\rangle \mu(dm) \tag{10}$$

where $\langle \cdot, \cdot \rangle$ is the Riemannian metric on $\mathcal{T}^n$ and μ is the Riemannian volume form.

Denote by β the subspace of $Vect^{(s)}$ consisting of all divergence-free vector fields. Then consider the orthogonal projection with respect to (10):

$$P : Vect^{(s)} \to \beta \tag{11}$$

It follows from the Hodge decomposition that the kernel of P is the subspace consisting of all gradients. Thus, for any $Y \in Vect^{(s)}$, we have

$$P(Y) = Y - grad p \tag{12}$$

where p is a certain H^{s+1}-function on $\mathcal{T}^n$, unique to within the additive constant for given Y.

Let a random flow $\xi(t, m)$ with initial data $\xi(0, m) = m \in \mathcal{T}^n$ be given on a flat n-dimensional torus $\mathcal{T}^n$ such that $\xi(t, m)$ is the general solution of a stochastic differential equation of the type (1). Suppose that $D_*\xi(t, m) = u(t, \xi(t, m))$, where $u(t, m)$ is a C^1-smooth in t and C^2-smooth in $m \in \mathcal{T}^n$ divergence-free vector field on $\mathcal{T}^n$. Suppose also that $\xi(t, m)$ satisfies the relation

$$PD_*D_*\xi(t, m) = F(t, m) \tag{13}$$

where $F(t, m)$ is a divergence-free vector field on $\mathcal{T}^n$. Taking into account formulae (8), (9) we obtain

$$PD_*D_*\xi(t, m) = P(\frac{\partial}{\partial t}u + (u, \nabla)u - \mathfrak{B}u) = \frac{\partial}{\partial t}u + (u, \nabla)u - \mathfrak{B}u - grad p \tag{14}$$

Thus (13) means that $u(t, m)$ is divergence-free and satisfies the relation

$$\frac{\partial}{\partial t}u + (u, \nabla)u - \mathfrak{B}u - grad p = F(t, m) \tag{15}$$

that is the Navier-Stokes type equation with the viscous term $\mathfrak{B}$ and external force $F(t, m)$. We interpret (13) as a stochastic analogue of Newton's second law on the group of Sobolev diffeomorphisms $D^s(\mathcal{T}^n)$ of the torus, subjected to the mechanical constraint.

3. BASIC NOTIONS FROM THE GEOMETRY OF GROUPS OF DIFFEOMORPHISMS OF FLAT TORUS

Here, following [1], [2], [6], we present the basic facts from the geometry on infinite-dimensional manifolds of Sobolev diffeomorphisms of the flat n-dimensional torus $\mathcal{T}^n$.

The tangent bundle to $\mathcal{T}^n$ is trivial: $T\mathcal{T}^n = \mathcal{T}^n \times \mathbb{R}^n$. Note that the flat metric generates in the second factor the inner product, same as in the copy of $\mathbb{R}^n$ from which the torus is obtained by the factorization.

Consider the set $\mathcal{D}^s(\mathcal{T}^n)$ of all diffeomorphisms of $\mathcal{T}^n$ to itself belonging to the Sobolev space H^s, where $s > n/2+1$. Recall that for $s > n/2+1$ the maps belonging to H^s are C^1-smooth.

There is a structure of smooth Hilbert manifold on $\mathcal{D}^s(\mathcal{T}^n)$ as well as the natural group structure with the respect to composition. A detailed descroption of structures and their interconnections can be found in [6].

The tangent space $T_e\mathcal{D}^s(\mathcal{T}^n)$ at unit $e = id$ is $Vect^{(s)}$. Also $T_e\mathcal{D}^s(\mathcal{T}^n)$ contains its subspace β consisting of all divergence-free vector fiels on $\mathcal{T}^n$ belonging to H^s.

The space $T_f\mathcal{D}^s(\mathcal{T}^n)$ where $f \in \mathcal{D}^s(\mathcal{T}^n)$ consists of the maps $Y : \mathcal{T}^n \to T\mathcal{T}^n$ such that $\pi Y(m) = f(m)$, where $\pi : T\mathcal{T}^n \to \mathcal{T}^n$ is a natural projection. For any $Y \in T_f\mathcal{D}^s(\mathcal{T}^n)$ there exists unique $X \in T_e\mathcal{D}^s(\mathcal{T}^n)$ such that $Y = X \circ f$. In any $T_f\mathcal{D}^s(\mathcal{T}^n)$ we can define the L^2-inner product by analogy with (10) by formula

$$(X, Y) = \int\limits_{\mathcal{T}^n} \langle X(m), Y(m)\rangle_{f(m)}\mu(dm) \tag{16}$$

The family of these inner products forms a weak Riemannian metric on $\mathcal{D}^s(\mathcal{T}^n)$.

The right translation $R_f : \mathcal{D}^s(T^n) \to \mathcal{D}^s(T^n)$, where $R_f(\Theta) = \Theta \circ f$ for $\Theta, f \in \mathcal{D}^s(T^n)$, is C^∞-smooth. The tangent to the right translation takes the form $TR_f(X) = X \circ f$ for $X \in T\mathcal{D}^s(T^n)$.

The left translation $L_f : \mathcal{D}^s(T^n) \to \mathcal{D}^s(T^n)$ where $L_f(\Theta) = f \circ \Theta$ for $\Theta, f \in \mathcal{D}^s(T^n)$ is only continuous. But if we specify a vector $x \in \mathbb{R}^n$ and denote by $l_x : T^n \to t^n$ the diffeomorphism $l_x(m) = m + x$ modulo factorization with respect to the integral lattice, we obtain C^∞-smooth left translation Ll_x.

Introduce the operators

$$B : T\mathcal{T}^n \to \mathbb{R}^n \tag{17}$$

the projection onto the second factor in $\mathcal{T}^n \times \mathbb{R}^n$, and

$$A(m) : \mathbb{R}^n \to T_m\mathcal{T}^n \tag{18}$$

the converse to B linear isomorphism of $\mathbb{R}^n$ onto the tangent space to $\mathcal{T}^n$ at $m \in \mathcal{T}^n$. The map A may be considered as a map $A : \mathbb{R}^n \to \beta \subset T_e\mathcal{D}^s(\mathcal{T}^n)$.

Introduce the linear isomorphism

$$Q_{g(m)} = A(g(m)) \circ B \tag{19}$$

where $g \in \mathcal{D}^s(\mathcal{T}^n)$, $m \in \mathcal{T}^n$. For every $Y \in T_f\mathcal{D}^s(\mathcal{T}^n)$ we get $Q_gY = A(g(m)) \circ B(Y(m)) \in T_g\mathcal{D}^s(\mathcal{T}^n)$ for any $f \in \mathcal{D}^s(\mathcal{T}^n)$. In particular, $Q_eY \in Vect^{(s)}$. The operation Q_e is a formalization for $\mathcal{D}^s(\mathcal{T}^n)$ of the usual finite-dimensional operation that allows one to consider the composition $X \circ f$ of a vector $X \in Vect^{(s)}$ and diffeomorphism $f \in \mathcal{D}^s(\mathcal{T}^n)$ as a vector in $T\mathcal{D}^s(\mathcal{T}^n)$. It denotes the shift of a vector, applied at point $f(x)$, to the point x with respect to global parallelism of the tangent bundle to torus.

Lemma 2. *The following relations hold:*

$$TR_{g^{-1}}(Q_gX) = Q_e(TR_{g^{-1}}X), \tag{20}$$

$$TR_g(Q_{g^{-1}}X) = Q_e(TR_gX). \tag{21}$$

Introduce the subspace $\beta_f \subset T_f\mathcal{D}^s(\mathcal{T}^n)$ as $TR_f\beta$ with β introduced in (11). Having done this for every $f \in \mathcal{D}^s(\mathcal{T}^n)$, we obtain the smooth subbundle $\bar{\beta}$ of $T\mathcal{D}^s(\mathcal{T}^n)$. Its integral manifold going through e is the submanifold and subgroup $\mathcal{D}^s_\mu(\mathcal{T}^n)$ that consists of H^s-diffeomorphisms preserving the volume.

Consider the map $\bar{P} : Vect^{(s)} \to \bar{\beta}$ determined for each $f \in \mathcal{D}^s(\mathcal{T}^n)$ by the formula

$$\bar{P}_f = TR_f \circ P \circ TR_f^{-1} \tag{22}$$

where $P = \bar{P}_e : Vect^{(s)} = T_e\mathcal{D}^s(\mathcal{T}^n) \to \beta = \beta_e = T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$ – the orthogonal projection introduced with respect to the Riemannian metric (12). It is shown in [6] that $\bar{P}$ is $T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$-right-invariant and $\mathcal{C}^\infty$-smooth.

It is a standard fact of differential geometry that the covariant derivative $\frac{\widetilde{D}}{dt}Y(t)$ of a vector filed $Y(t)$ along a curve $g(t)$ in $\mathcal{D}^s_\mu(\mathcal{T}^n)$ is defined by the relation

$$\frac{\widetilde{D}}{dt}Y(t) = \bar{P}\frac{\bar{D}}{dt}Y(t) \tag{23}$$

where $\frac{\bar{D}}{dt}$ is the covariant derivative on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ generated by the derivative on $\mathcal{T}^n$ (see [6]). Let $F(t,\bar{g},Y)$, $Y \in T_g\mathcal{D}^s_\mu(\mathcal{T}^n)$, be a force vector field on $\mathcal{D}^s_\mu(\mathcal{T}^n)$. Consider a curve $g(t)$ satisfying the equation

$$\frac{\widetilde{D}}{dt}\dot{g}(t) = \bar{F}(t, g(t), \dot{g}(t)) \tag{24}$$

Denote by $u(t)$ the curve in $T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$ (i.e., a divergence-free vector field on $\mathcal{T}^n$) obtained by right translations of vectors $\dot{g}(t)$, i.e., $u(t) = \dot{g}(t) \circ g^{-1}(t) = TR^{-1}_{g(t)}\dot{g}(t)$.

The field $u(t)$ satisfies the Euler equation

$$\frac{\partial}{\partial t} + (u \cdot \nabla)u - gradp = TR^{-1}\bar{F}(t, g(t), u(t, g(t))). \tag{25}$$

It should be pointed out that (24) is Newton's second law with the force $\bar{F}$ that describes the motion of perfect incompressible fluid on $\mathcal{T}^n$ under the action of force $TR^{-1}\bar{F}(t, g(t), u(t, g(t)))$ depending of the "configuration of fluid". Recall that a curve satisfying (25) with $\bar{F} = 0$ is called a geodesic.

If $\bar{F}$ is a right-invariant vector field on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ such that $\bar{F}_e = F$, where F is a divergence-free vector field on $\mathcal{T}^n$, then (25) turns into

$$\frac{\partial}{\partial t} + (u \cdot \nabla)u - gradp = F. \tag{26}$$

Consider the map $\bar{A} : \mathcal{D}^s_\mu(\mathcal{T}^n \times \mathbb{R}^n \to T\mathcal{D}^s_\mu(\mathcal{T}^n)$ such that $\bar{A}_e$ is equal to A introduced earlier, and for every $g \in \mathcal{D}^s_\mu(\mathcal{T}^n)$ the map $\bar{A}_g : \mathbb{R}^n \to T_g\mathcal{D}^s_\mu(\mathcal{T}^n)$ is obtained from $\bar{A}_e$ by means of the right translation, i.e., for $X \in \mathbb{R}^n$:

$$\bar{A}_g(X) = TR_g \circ A_e(X) = (A \circ g)(X). \tag{27}$$

The right invariant vector field $\bar{A}(X)$ is C^∞-smooth on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ for every specified $X \in \mathbb{R}^n$.

For any point $m \in \mathcal{T}^n$ denote by $exp_m : T_m\mathcal{T}^n \to \mathcal{T}^n$ the map that sends the vector $X \in T_m\mathcal{T}^n$ into the point $m+X$ in $\mathcal{T}^n$, where $m+X$ is obtained modulo factorization with respect to integral lattice, i.e. by the following procedure: we take a certain point $\mathbb{R}^n$ corresponding to $m \in \mathcal{T}^n$ and $X \in \mathbb{R}^n = T_m\mathbb{R}^n$ then we identify $\mathbb{R}^n$ with $T_m\mathbb{R}^n = \mathbb{R}^n$, find $m + X$ in $\mathbb{R}^n$ and pass from $\mathbb{R}^n$ to $\mathcal{T}^n$ by factorization with respect to $\mathbb{Z}^n$. The field of maps exp at all points generates the map $\overline{exp} : T_e\mathcal{D}^s(\mathcal{T}^n) \to \mathcal{D}^s(\mathcal{T}^n)$ that sends the vector $X \in T_e\mathcal{D}^s(\mathcal{T}^n)$ to $e + X \in \mathcal{D}^s(\mathcal{T}^n)$, where $e + X$ is the diffeomorphism of $\mathcal{T}^n$ of the form $(e + X)(m) = m + X(m)$.

Consider the composition $\overline{exp} \circ \bar{A}_e : \mathbb{R}^n \to \mathcal{D}^s(\mathcal{T}^n)$. By the construction of $\bar{A}_e$ for any $X \in \mathbb{R}^n$ we get $\overline{exp} \circ \bar{A}_e(X)(m) = m + X$, i.e., the same vector X is added to every point m. Thus, obviously, $\overline{exp} \circ \bar{A}_e(X) \in \mathcal{D}^s_\mu(\mathcal{T}^n)$.

Let $w(t)$ be a Wiener process in $\mathbb{R}^n$ defined on a certain probability space $(\Omega, \mathcal{F}, P)$. Introduce the process

$$W(t) = \overline{exp} \circ \bar{A}_e(\mathbf{B}w(t)) \tag{28}$$

in $\mathcal{D}^s_\mu(\mathcal{T}^n)$, where $\mathbf{B}$ is the constant linear operator. By the construction, for $\omega \in \Omega$ the corresponding sample trajectory $W_\omega(t)$ is the diffeomorphism of the form $W_\omega(t)(m) = m + \mathbf{B}w_\omega(t)$, so that the same sample trajectory $\mathbf{B}w_\omega(t)$ is added to each point $m \in \mathcal{T}^n$. In particular, this clarifies that $W_\omega(t)(m)$ takes values in $\mathcal{D}^s_\mu(\mathcal{T}^n)$. Note that for a specified $\omega \in \Omega$ and for specified $t \in \mathbb{R}$ the value of $\mathbf{B}w_\omega(t)$ is a constant vector in $\mathbb{R}^n$. Then for given ω and t, the action of $W_\omega(t)$ coincides with that of $l_{\mathbf{B}w_\omega(t)}$.

4. VISCOUS INCOMPRESSIBLE FLUIDS

Consider $s > n/2+1$, so that the diffeomorphisms from $\mathcal{D}^s(\mathcal{T}^n)$ and so $\mathcal{D}^s_\mu(\mathcal{T}^n)$ are $\mathcal{C}^1$-smooth and $Vect^{(s)}$ consists of $\mathcal{C}^2$-smooth vector fields. Everywhere below we use the same process $W(t)$ constructed from the specified Wiener process $w(t)$ in $\mathbb{R}^n$ by formula (28).

Let $g(t)$ be a solution of (24) on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ with $\bar F = 0$ and with initial conditions $g(0) = e$ and $\dot g(0) = u_0 \in T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$. Such a solution exists in a certain time interval $t \in [0,T]$. Consider $u(t) = \dot g(t) \circ g^{-1}(t) \in T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$. This infinite-dimensional vector considered as a divergence free vector field on $\mathcal{T}^n$, will be denoted $u(t,m)$.

Consider a process on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ of the form $\eta(t) = W(t)\circ g(t)$. In finite-dimensional notation $\eta(t)$ is a random diffeomorphism of $\mathcal{T}^n$ of the form $\eta(t,m) = g(t,m) + \mathbf{B}w(t)$ modulo the factorization with respect to integral lattice. Introduce the process $\xi(t) = \eta(T-t)$, or, in finite-dimensional notation, $\xi(t,m) = g(T-t,m) + \mathbf{B}w(T-t)$. Since $w(t)$ is a martingale with respect to its own "past" , we can derive from the properties of conditional expectation that $D_*\xi(t) = \dot g(T-t,m) = u(T-t, g(T-t,m))$, and so $\bar P D_* D_* \xi(t) = \frac{\bar D}{ds}\dot g(s)|_{s=T-t} = 0$ on $\mathcal{D}^s_\mu(\mathcal{T}^n)$.

Consider the random process

$$\xi_t(s) = \xi(s)\circ\xi^{-1}(t) = W(T-s)\circ g(T-s)\circ g^{-1}(T-t)\circ (W(T-t))^{-1}.$$

Notice that the random diffeomorphism $(W(T-t))^{-1}$ acts by the rule $(W(T-t))^{-1}(m) = m - \mathbf{B}w(t)$. Thus

$$\xi_t(t) = \xi(t)\circ\xi^{-1}(t) = W(T-t)\circ g(T-t)\circ g^{-1}(T-t)\circ (W(T-t))^{-1} = e.$$

The finite-dimensional descroption of this process can be given as follows. By the construction

$$m = \xi(t, \xi^{-1}(t,m)) = g(T-t, \xi^{-1}(t,m)) + \mathbf{B}w(t).$$

Then

$$g(T-t, \xi^{-1}(t,m)) = m - \mathbf{B}w(t)$$

and so

$$\xi^{-1}(t,m) = g^{-1}(T-t, m - \mathbf{B}w(t)).$$

Thus

$$\xi_t(s,m) = \xi(s, g^{-1}(T-t, m-\mathbf{B}w(t))) = g(T-s, g^{-1}(T-t, m-\mathbf{B}w(t))) + \mathbf{B}w(t).$$

Notice that $\xi_t(t,m) = m - \mathbf{B}w(t) + \mathbf{B}w(t) = m$, i.e., $\xi_t(t) = e$ on $\mathcal{D}^s_\mu(\mathcal{T}^n)$. Then the "present" σ-algebra $\mathcal{N}^\xi_t$ is trivial and so the conditional expectation with the respect to it coincides with ordinary mathematical expectation. Hence, using the relation between $u(t)$ and $g(t)$ and definition of D_* we can easily derive that

$$D_*\xi_t(s) = E(u(T-t, m - \mathbf{B}w(T-t))) = E(Q_e TR^{-1}_{W(T-t)} u(T-t)). \tag{29}$$

Introduce on $\mathcal{T}^n$ the vector field $U(t,m) = E(u(t, m-\mathbf{B}w(t)))$ (in infinite-dimensional notation $U(t) = E(Q_e TR^{-1}_{W(t)} u(t))$).

Lemma 3 (See [1]). *The vector field $U(t,m)$ is divergence free.*

Proof. By construction, for an elementary event $\omega \in \Omega$, the diffeomorphism $(W(t)_\omega)^{-1}$ is a shift of the entire torus by a constant vector. Hence, Q_e applied to $TR^{-1}_{W(t)_\omega} u(t)$ means the parallel translation on torus of the entire divergence free vector field $u(t)$ by the same constant vector back. Thus $Q_e TR^{-1}_{W(t)} u(t)$ is a random divergence free vector field on the torus. Hence its expectation is divergence free. □

So, $U(t) \in T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$. In particular, we have proved above that $D_*\xi_t(s)_{|s=t} = U(T-t)$.

Theorem 1. *Vector field $U(t,m)$ satisfies the following Reynolds type equation*

$$\frac{\partial}{\partial t}U + E\left[((u\cdot\nabla)(t, m - \mathbf{B}w(t)))\right] - \widetilde{\mathfrak{B}}U - gradp = 0. \tag{30}$$

Proof. It follows from Itô formula that

$$u(t, m-\mathbf{B}w(t)) = \int_0^t \frac{\partial u}{\partial s}(s, m-\mathbf{B}w(s))ds + \int_0^t \widetilde{\mathfrak{B}}u ds - \int_0^t u'\,\mathbf{B}\,dw(s),$$

where u' is the linear operator of derivative of u in $m \in \mathcal{T}^n$. Recall that $u(t,m)$ satisfies the Euler equation without external force, i.e., $\frac{\partial u}{\partial t} = -P((u\cdot\nabla)u)$. Since $\frac{\partial}{\partial t}Eu(t, m-\mathbf{B}w(t)) = \frac{\partial}{\partial t}U(t)$ and $E\big(\int_0^t u'\,\mathbf{B}dw(t)\big) = 0$, we derive that

$$\frac{\partial}{\partial t}U = \frac{\partial}{\partial t}E\left(u(t, m-\mathbf{B}w(t))\right) = PE\Big(-(u\cdot\nabla)u(t, m-\mathbf{B}w(t)) + \widetilde{\mathfrak{B}}u(t, m-\mathbf{B}w(t))\Big) =$$

$$-E\Big((u\cdot\nabla)u(t, m-\mathbf{B}w(t))\Big) + \widetilde{\mathfrak{B}}U + \operatorname{grad} p.$$

So, (30) is satisfied. □

There are usual methods for transforming (30) into the standard Reynolds form. For a divergence-free vector field $X(m)$ on $\mathcal{T}^n$ introduce the random divergence-free vector field

$$\breve{U}_X(t,m) = X(m-\mathbf{B}w(t)) - E(X(m-\mathbf{B}w(t)))$$

or in the infinite-dimensional notation

$$\breve{U}(t) = Q_e TR^{-1}_{W(T-t)}X - E(Q_e TR^{-1}_{W(T-t)}X).$$

For $X = u(t)$ we obtain

$$\breve{U}_u(t,m) = u(m-\mathbf{B}w(t)) - E(u(m-\mathbf{B}w(t))) = u(t, m-\mathbf{B}w(t)) - U(t,m);$$

$$E(\breve{U}_u(t,m)) = E(u(m-\mathbf{B}w(t))) - E(E(u(m-\mathbf{B}w(t)))) = E(u(m-\mathbf{B}w(t))) - E(u(m-\mathbf{B}w(t))) = 0,$$

and so

$$u(t, m-\mathbf{B}w(t)) = U(t,m) + \breve{U}_u(t,m). \tag{31}$$

From (31) we can see

$$E\left[((u\cdot\nabla)(t, m-\mathbf{B}w(T)))\right] = (U\cdot\nabla)U + E\left[(\breve{U}_{u(t)}\cdot\nabla)\breve{U}_{u(t)}\right].$$

Thus (30) transforms into

$$\frac{\partial}{\partial t}U + (U\cdot\nabla)U - \widetilde{\mathfrak{B}}U - gradp = -E\left[(\breve{U}_{u(t)}\cdot\nabla)\breve{U}_{u(t)}\right], \tag{32}$$

which is the analogue of standard form of Reynolds equation with the viscous term $\mathfrak{B}U$. It differs from the Navier-Stokes type relation with such viscous term by the external force $-E\left[(\breve{U}_{u(t)} \cdot \nabla)\breve{U}_{u(t)}\right]$ that depends on $u(t,m)$, not on $U(t,m)$.

We can show that a slight modification of the above scheme of arguments allows us to annihilate the external force in (32) by introducing a special random force field on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ into (24).

For a random divergence free a.s. $H^{s+1}-$vector field $X_\omega(m)$ on $\mathcal{T}^n$ (i.e., for a random vector $X_\omega \in T_e\mathcal{D}^{s+1}_\mu(\mathcal{T}^n) \subset T_e\mathcal{D}^s_\mu(\mathcal{T}^n))$), construct the random vector field $\breve{U}_{X_\omega}(t,m)$ which for any $\omega \in \Omega$ is given by the formula

$$\breve{U}_{X_\omega}(t,m) = X_\omega(m - \mathbf{B}w_\omega(t)) - E(X_\omega(m - \mathbf{B}w_\omega(t))).$$

Introduce the non-random H^s vector field $PE\left[(\breve{U}_{X_\omega} \cdot \nabla)\breve{U}_{X_\omega}\right]$ and then construct the random vector $\mathfrak{F}_\omega(t,X_\omega)$ in $T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$ by the formula

$$\mathfrak{F}_\omega(t,X_\omega) = Q_eTR_{W_\omega(t)}PE\left[(\breve{U}_{X_\omega} \cdot \nabla)\breve{U}_{X_\omega}\right].$$

Note that $PE\left[(\breve{U}_{X_\omega} \cdot \nabla)\breve{U}_{X_\omega}\right]$ and so $\mathfrak{F}_\omega(t,X_\omega)$ lose the derivatives, i.e., they are H^s-vector fields only since X_ω is H^{s+1}. Thus $\mathfrak{F}_\omega(t,X_\omega)$ is well-posed only on an everywhere dense subset $T_e\mathcal{D}^{s+1}_\mu(\mathcal{T}^n)$ in $T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$.

Now introduce the right-invariant force vector field $\bar{\mathfrak{F}}_\omega(t,g,Y_\omega)$, where $Y_\omega \in T_g\mathcal{D}^s_\mu(\mathcal{T}^n)$ on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ that at $g \in \mathcal{D}^{s+1}_\mu(\mathcal{T}^n)$ and $\omega \in \Omega$ is determined by the formula

$$\bar{\mathfrak{F}}_\omega(t,g,Y_\omega) = TR_g\mathfrak{F}_\omega(t,TR_g^{-1}Y_\omega)$$

where $TR_g^{-1}Y_\omega$ is a divergence free a.s. $H^{s+1}-$vector field.

Consider the equation

$$\frac{\bar{D}}{dt}\dot{g}_\omega(t) = \bar{\mathfrak{F}}_\omega(t,g_\omega(t),\dot{g}_\omega(t)) \tag{33}$$

on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ whose right hand side is well-posed on the everywhere dense subset $\mathcal{D}^{s+1}_\mu(\mathcal{T}^n)$ in $\mathcal{D}^s_\mu(\mathcal{T}^n)$. Note that (33) has no diffusion term, so it's an ordinary differential equation with parameter $\omega \in \Omega$.

Suppose that for the initial condition $g_\omega(0) = e$ and $\dot{g}(0) = u_0 \in T_e\mathcal{D}^{s+1}_\mu(\mathcal{T}^n)$ it has a unique $H^{s+1}-$solution g_ω which is a.s. well-posed on a non-random time interval $t \in [0,T]$. Consider the divergence free a.s. $H^{s+1}-$vector field $u_\omega(t,m)$ on $\mathcal{T}^n$ given by the relation $u_\omega(t,m) = \dot{g}_\omega(t,m) \circ g_\omega^{-1}(t,m)$. The analogue of above-mentioned vector U now takes the form

$$\mathbb{U}(t,m) = E(u_\omega(t,m - \mathbf{B}w_\omega(t))). \tag{34}$$

As well as vector field $U(t,m)$, this vector field is divergence free.

Theorem 2. *The divergence free vector field $\mathbb{U}$ given by (34), satisfies the analogue of Navier-Stokes equation with viscousity viscosity $\mathfrak{B}$ and without external force:*

$$\frac{\partial}{\partial t}\mathbb{U} + (\mathbb{U} \cdot \nabla)\mathbb{U} - \mathfrak{B}\mathbb{U} - gradp = 0$$

where $\mathfrak{B}$ is the second order differential operator $\mathfrak{B} = \frac{1}{2}\widetilde{\mathfrak{B}}^{ij}\frac{\partial}{\partial x^i \partial x^j}$, $(\widetilde{\mathfrak{B}}^{ij}) = \mathbf{B}\mathbf{B}^*$.

Proof.

Since

$$\mathbb{U}(t,m) = E(u_\omega(t,m - \mathbf{B}w_\omega(t))),$$

and

$$E(du_\omega(t, m - \mathbf{B}w_\omega(t))) = E(\frac{\partial}{\partial t}u_\omega(t, m - \mathbf{B}w_\omega(t)) + \mathfrak{B}u_\omega(t, m - \mathbf{B}w_\omega(t))) =$$
$$= E(-P\left[((u_\omega \cdot \nabla)u_\omega)\right] + \mathfrak{F}_\omega(t, u_\omega(t)) + \mathfrak{B}u_\omega(t, m - \mathbf{B}w_\omega(t))),$$

we get

$$\frac{\partial}{\partial t}\mathbb{U} = E(\frac{\partial}{\partial t}u_\omega(t, m - \mathbf{B}w_\omega(t))) =$$
$$= -E\left[((u_\omega \cdot \nabla)u_\omega)(t, m - \mathbf{B}w_\omega(t))\right] + \mathfrak{B}\mathbb{U} + gradp + EQ_eTR^{-1}_{W(t)}\mathfrak{F}_\omega(t, u_\omega(t)) =$$
$$= -(\mathbb{U}\cdot\nabla)\mathbb{U} + \mathfrak{B}\mathbb{U} + gradp - E\left[(\breve{U}_{u_\omega(t)} \cdot \nabla)\breve{U}_{u_\omega(t)}\right] + EQ_eTR^{-1}_{W(t)}\mathfrak{F}_\omega(t, u_\omega(t)) =$$
$$= -(\mathbb{U}\cdot\nabla)\mathbb{U} + \mathfrak{B}\mathbb{U} + gradp - E\left[(\breve{U}_{u_\omega(t)} \cdot \nabla)\breve{U}_{u_\omega(t)}\right] + PE\left[(\breve{U}_{u_\omega(t)} \cdot \nabla)\breve{U}_{u_\omega(t)}\right]. \quad (35)$$

Recall that for the divergence free fields $\breve{U}$, the vector fields $\frac{\partial}{\partial t}\breve{U}$ is divergence free. From definition (12) of operator P we obtain

$$PE\left[(\breve{U}_{u_\omega(t)} \cdot \nabla)\breve{U}_{u_\omega(t)}\right] = E\left[(\breve{U}_{u_\omega(t)} \cdot \nabla)\breve{U}_{u_\omega(t)}\right] - gradp.$$

Introduce p_1 and p_2 by relations $P((\mathbb{U}\cdot\nabla)\mathbb{U} + \mathfrak{B}\mathbb{U}) = (\mathbb{U}\cdot\nabla)\mathbb{U} + \mathfrak{B}\mathbb{U} - gradp_1$ and $PE\left[(\breve{U}_{u_\omega(t)} \cdot \nabla)\breve{U}_{u_\omega(t)}\right] = E\left[(\breve{U}_{u_\omega(t)} \cdot \nabla)\breve{U}_{u_\omega(t)}\right] - gradp_2$.

Thus, using these relations we obtain (34) from (35), so $\frac{\partial}{\partial t}\mathbb{U} + (\mathbb{U}\cdot\nabla)\mathbb{U} - \mathfrak{B}\mathbb{U} - gradp = 0$.

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