UDC 517.9

ON SOME SPECIAL TYPES OF ε-APPROXIMATIONS FOR SET-VALUED MAPPINGS

Yuri E. Gliklikh

Voronezh State University

Поступила в редакцию 23.03.2009 г.

Аннотация. Для полунепрерывных сверху конечномерных многозначных отображений с выпуклыми или асферичными замкнутыми значениями мы доказываем существование специальных непрервных ε-аппроксимаций, которые поточечно сходятся к измеримому по Борелю селектору многозначного отображения при ε, стремящимся к нулю. Для выпуклозначного случая сходимость имеет место на всей области определения, а для отображений с асферичными значениями — на некотором счетном всюду плотном подмножестве.

Ключевые слова: Полунепрерывные сверху многозначные отображения; выпусклые замкнутые значения; асферичные замкнутые значения; ε-аппроксимации; поточечная сходимость.

Abstract. For upper semicontinuous finite-dimensional set-valued mappings with either convex closed or aspheric closed values we prove the existence of special continuous ε-approximations that point-wise converge to a Borel measurable selector of the set-valued mapping as ε tends to zero. For convex-valued case the convergence holds on the entire domain while for aspheric-valued case — on a certain countable everywhere dense subset.

Key words: Upper semicontinuous set-valued mapping; convex closed values; aspheric closed values; ε-approximations; point-wise convergence.

1. INTRODUCTION

The main aim of this paper is to show the existence of ε-approximations of the upper semicontinuous set-valued finite-dimensional map that point-wise converge to a Borel measurable selector as $\varepsilon \to 0$. Unlike the case of ordinary differential inclusions, such approximations are very much useful for investigation of stochastic differential inclusions.

Recall that ε-approximations are proved to exist for upper semicontinuious set-valued map either with convex closed values or with aspheric closed ones (see below). We consider both cases, but for convex-valued mappings we prove the existence of point-wise converging ε-approximations on the entire domain while for aspheric-valued ones only on a certain countable everywhere dense subset.

The paper is partially a survey of some results from [1,2] (convex-valued case) and partially it contains new results (aspheric-valued case).

The structure of paper is as follows. In Section 2 we give a short introduction into the Theory of Set-Valued Mappings. More details can be found, e.g., [3,4] where in particular the proofs of many results, presented here, are given.

In Section 3 we deal with convex-valued mappings. Taking into account applications to stochastic differential inclusions, we consider two classes of set-valued mappings: those depending on points of phase space and those depending on curves but non-anticipating with respect to a special filtration generated by σ-algebras of cylinder sets. We prove the existence of point-wise converging sequences of ε-approximation depending on points or non-anticipating with respect to the same filtration, respectively.

In Section 4 we construct a sequence of ε-approximations for aspheric-valued mappings that point-wise converge to a selector on a countable everywhere dense subset Ξ. In this case for every point of Ξ there exists a number such that for all greater numbers the values of all terms of the sequence at that point are stabilized (i.e., have the same value).

The research is supported in part by RFBR Grants 07-01-00137 and 08-01-00155.

2. A BRIEF INTRODUCTION INTO THE THEORY OF SET-VALUED MAPPINGS

A set-valued mapping F from a set X into a set Y is a correspondence that assigns a non-empty subset $F(x) \subset Y$ to every point $x \in X$; $F(x)$ is called the value of x.

In order to distinguish set-valued mappings form single-valued ones we shall denote a set-valued mapping F sending X to Y, by the symbol $F : X \multimap Y$ while for a single-valued mapping we shall keep the notation $f : X \to Y$.

If X and Y are metric spaces, for set-valued mappings there are several different analogues of continuity that in the case of single-valued mappings are transformed into the usual one (here we do not deal with the description of such notion for set-valued mappings of topological spaces, see, e.g., [3]).

Definition 1. *A set-valued mapping F is called upper semicontinuous at the point $x \in X$ if for each $\varepsilon > 0$ there exists a neighbourhood $U(x)$ of x such that from $x' \in U(x)$ it follows that $F(x')$ belongs to the ε-neighbourhood of the set $F(x)$. F is called upper semicontinuous on X if it is upper semicontinuous at every point of X.*

Definition 2. *A set-valued mapping F is called lower semicontinuous at the point $x \in X$ if for each $\varepsilon > 0$ there exists a neighbourhood $U(x)$ of x such that from $x' \in U(x)$ it follows that $F(x)$ belongs to the ε-neighbourhood of $F(x')$. F is called lower semicontinuous on X if it is lower semicontinuous at every point of X.*

Definition 3. *If F is both upper and lower semicontinuous, it is called continuous (sometimes is also called Hausdorff continuous).*

The continuous set-valued mapping F such that for each x its value $F(x)$ is a closed bounded set, are continuous with respect to the so called Hausdorff metric on the space of all non-empty closed bounded subsets in Y. In order to describe it we first introduce the submetric $\bar{H}(A,B) = \sup_{a \in A} \rho(a,B)$ where ρ is the metric in Y. Then the Hausdorff metric is defined by the formula

$$H(A,B) = \max(\bar{H}(A,B), \bar{H}(B,A)). \qquad (1)$$

A set-valued mapping is called closed if its graph is a closed subset in $X \times Y$. If F is closed and for each point $x \in X$ there exists a neighbourhood $U(x)$ such that $F(U(x))$ is relatively compact, F is upper semicontinuous.

Definition 4. *We say that $F(t,x)$ satisfies upper Carathéodory conditions if:*

1) for every $x \in X$ the map $F(\cdot,x) : I \multimap Y$ is measurable,

2) for almost all $t \in I$ the map $F(t,\cdot) : X \multimap Y$ is upper semicontinuous.

Definition 5. *Let $I = [0,l] \subset R$. The set-valued mapping $F : I \times X \multimap Y$ is called almost lower semicontinuous if there exists a countable sequence of disjoint compact sets $\{I_n\}$, $I_n \subset I$ such that: (i) the measure of $I \setminus \cup_n I_n$ is equal to zero; (ii) the restriction of F on each $I_n \times X$ is lower semicontinuous.*

An important technical role in investigating set-valued mappings is played by single-valued mappings that approximate the set-valued ones in some sense. We describe two kinds of such single-valued mappings: selectors and ε-approximations.

Definition 6. *Let $F : X \multimap Y$ be a set-valued mapping. A single-valued mapping $f : X \to Y$ such that for each $x \in X$ the inclusion $f(x) \in F(x)$ holds, is called a selector of F.*

Not every set-valued mapping has a continuous selector. For lower semicontinuous set-valued mappings with convex closed values their existence is proved in the classical Michael's Theorem.

Theorem 7. (Michael's Theorem) *If X is an arbitrary metric space and Y is a Banach space, a lower semicontinuous mapping such that the value of every point of X is a convex closed set, has a continuous selector.*

If the values of a lower semicontinuous set-valued mapping (generally speaking) are not convex, it may not have continuous selectors. Then the following construction is often very much useful.

Definition 8. *Let E be a separable Banach space. A non-empty set $\mathcal{M} \subset L^1([0,l];E)$ is called decomposable if $f * \chi_m + g * \chi_{[0,l]\setminus m} \in \mathcal{M}$ for all $f, g \in \mathcal{M}$ and for every measurable subset m in $[0,l]$ where χ is the characteristic function of the corresponding set.*

The reader can find more details about decomposable sets in [4] and [5].

Theorem 9. (Bressan—Colombo Theorem) *Consider a separable metric space (Ω, d). Let X be a Banach space and $(J, \mathcal{A}, \mu)$ be a measurable space with a σ-algebra $\mathcal{A}$ and a non-atomic measure μ such that $\mu(J) = 1$. Consider the space $Y = L^1_X(J, \mathcal{A}, \mu)$ of integrable mappings from $(J, \mathcal{A}, \mu)$ into X. If a set-valued mapping $F : \Omega \multimap Y$ is lower semicontinuous and has close decomposable values, F has a continuous selector.*

The assertion of Theorem 9 is proved, e.g., as Lemma 9.2 in [5].

Upper semicontinuous mappings arise in applications more often than lower semicontinuous ones. Generally speaking, they do not have continuous selectors (but they have measurable ones). The so called ε-approximations are very much useful for investigating the upper semicontinuous mappings.

Definition 10. *For given $\varepsilon > 0$ a continuous single-valued mapping $f_\varepsilon : X \to Y$ is called an ε-approximation of a set-valued mapping $F : X \multimap Y$ if the graph of f as a set in $X \times Y$, belongs to the ε-neighbourhood of the graph of F.*

We mention the following classes of upper semicontinuous set-valued mappings of finite-dimensional spaces, for which the existence of ε-approximations is proved for each $\varepsilon > 0$:

— the mappings with convex closed values;

— the so called mappings with values that are aspheric in all dimensions from 1 to $n-1$ and weakly aspheric in the dimension n (see [6]). This class of of set-valued mappings was first time considered by A. D. Myshkis in 1954 [7]. In [6] and [8] topological characteristics of topological index and Lefschetz number types were constructed for such mappings. Later (in 80-th years of XX century) this class was rediscovered and called "the mappings whose values at every point have the so-called uv^k-property for $k = 1, \dots, n$" (see exact definition, e.g., in [9]).

Let X be a Banach space and $F : X \multimap X$ be an upper semicontinuous set-valued mapping with convex closed values. Let also for each bounded subset $\Omega \subset X$ its image $F(\Omega)$ is relatively compact. Then if F sends a ball B of X into itself, in B there exists a fixed point $x \in F(x)$ of F (an analog of Schauder's principle known as Glicksberg-Ky Fan Theorem).

Let $F : R \times R^n \multimap R^n$ be a set-valued mapping. A differential inclusion

$$\dot{x} \in F(t, x) \qquad (2)$$

is an analogue of differential equation and transforms into the latter if F is single-valued.

A solution of (2) is an absolutely continuous curve $x(t)$ such that (2) is satisfied for it almost everywhere.

If F is upper semicontinuous and has convex closed bounded values, for each couple $x_0 \in R^n$, $t_0 \in R$ there exists a local in time solution of (2) with the initial condition $x(t_0) = x_0$. It is also known that for an upper semicontinuous F with closed bounded (not necessarily convex) values there exists a solution of Cauchy problem for the differential inclusion

$$\dot{x} \in \overline{co}F(t, x),$$

where $\overline{co}F(t, x)$ is the convex closure of $F(t, x)$.

Existence of solutions of (2) for lower semicontinuous F is possible also for non-covex values. Often such existence can be proved by applying Bressan-Colombo Theorem 9.

3. SPECIAL ε-APPROXIMATIONS FOR CONVEX-VALUED UPPER SEMI-CONTINUOUS MAPPINS

Here, following [1, 2], we prove existence of special ε-approximations for upper semicontinuous mappings in finite-dimensional spaces with convex closed values such that they point-wise converge to a Borel measurable selector of the set-valued mapping as $\varepsilon \to 0$.

Theorem 11. *Let $\Phi : \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper semi-continuous set-valued map with convex closed bounded values. For a sequence $\varepsilon_i \to 0$ there exists a sequence of continuous ε_i approximations for Φ that point-wise converges to a Borel measurable selector of Φ. If Φ takes values in a convex set Ξ in $\mathbb{R}^n$, those ε-approximations take values in Ξ as well.*

Proof. It is shown in [10] that in the case under consideration for any ε_i there exists a lower semi-continuous set-valued map $\Psi_i : \mathbb{R}^n \multimap \mathbb{R}^n$ with closed convex bounded values such that: (i) for any $x \in \mathbb{R}^n$ the inclusion $\Phi(x) \subset \Psi_i(x)$ holds and (ii) the graph of Ψ_i belongs to the ε_i-neighbourhood of the graph of Φ. From the construction it follows that if Φ takes values in a convex set Ξ in $\mathbb{R}^n$, then the values of all $\Psi_i(x)$ belong to Ξ. Notice that for an upper semi-continuous mapping with compact values the sum of such mappings and the products with a continuous function are upper semi-continuous. Hence, from the proof of Theorem 2 [10] it follows that in the case under consideration all Ψ_i are continuous set-valued mapping and, in particular in our case, they are continuous with respect to Hausdorff metric.

Consider the minimal selector $\psi_i(\cdot)$ of $\Psi_i(\cdot)$, i.e., $\psi_i(x)$ is the closest to origin point in $\Psi_i(x)$, $x \in \mathbb{R}^n$. We refer the reader to [11] for complete description of minimal selectors. In particular, it is shown there that minimal selectors are continuous. Thus, ψ_i is an ε_i-approximation of Φ.

Specify an arbitrary point $x \in \mathbb{R}^n$. Since $\Phi(x) \subset \Psi_i(x)$ for each i, for the Hausdorff submetric $\bar{H}$ we have $\bar{H}(\Phi(x), \Psi_k(x)) = 0$. Hence for the Hausdorff metric H we obtain that $H(\Psi_i(x),\Phi(x)) = \bar{H}(\Psi_i(x),\Phi(x))$ for each i.

Now specify ε_k. By definition of upper semi-continuity for any $x \in \mathbb{R}^n$ there exists $\delta_k > 0$ such that for any x' from δ_k-neighbourhood of x the value $\Phi(x')$ belongs to the ε_k-neighbourhood of $\Phi(x)$. Since $\varepsilon_i \to 0$, $\varepsilon_{k+l} < \delta_k$ for some $l = l(k,x)$ and without loss of generality we may take $l(k,x) \geq 0$. Thus $\bar{H}(\Phi(x'),\Phi(x)) < \varepsilon_k$ for each x' from ε_{k+l}-neighbourhood of x.

Since the graph of Ψ_{k+l} belongs to ε_{k+l}-neighbourhood of the graph of Φ, there exists a point x'' in the ε_{k+l}-neighbourhood of x such that $\Psi_{k+l}(x)$ belongs to ε_{k+l}-neighbourhood of $\Phi(x'')$, i.e., $\bar{H}(\Psi_{k+l}(x),\Phi(x'')) < \varepsilon_{k+l}$.

Thus

$$H(\Psi_{k+l}(x),\Phi(x)) = \bar{H}(\Psi_{k+l}(x),\Phi(x)) \leq$$
$$\leq \bar{H}(\Psi_{k+l}(x),\Phi(x'')) + \bar{H}(\Phi(x''),\Phi(x)) <$$
$$< \varepsilon_{k+l} + \varepsilon_k < 2\varepsilon_k.$$

Hence at each x the convex set $\Psi_i(x)$ tends to the convex set $\Phi(x)$ with respect to Hausdorff metric as $i \to \infty$. Then $\psi_i(x)$ tends to the point $\varphi(x) \in \Phi(x)$ that is the closest to the origin. The fact that the point-wise limit $\varphi(\cdot)$ of the sequence of continuous mappings $\psi_i(\cdot)$ is a Borel measurable mapping, completes the proof. ■

Introduce $\tilde{\Omega} = C^0([0,T],\mathbb{R}^n)$, the Banach space of continuous curves in $\mathbb{R}^n$ given on $[0,T]$, with usual uniform norm, and the σ-algebra $\tilde{\mathcal{F}}$ on $\tilde{\Omega}$ generated by cylinder sets. By $\mathcal{P}_t$ we denote the σ-subalgebra of $\mathcal{F}$ generated by cylinder sets with bases over $[0,t] \subset [0,T]$. Recall that $\tilde{\mathcal{F}}$ is the Borel σ-algebra on $\tilde{\Omega}$ (see [12]).

Let $B:[0,T]\times\tilde{\Omega} \to Z$ be a mapping to some metric space Z. Below we shall often suppose that such mappings with various spaces Z satisfy the following condition:

Condition 12. *For each $t \in [0,T]$ from the fact that the curves $x_1(\cdot)$, $x_2(\cdot) \in \tilde{\Omega}$ coincide for $0 \leq s \leq t$, it follows that $B(t,x_1(\cdot)) = B(t,x_2(\cdot))$.*

Remark 13. *Note that the fact that a mapping B satisfies Condition 12, is equivalent to the fact that B at each t is measurable with respect to Borel σ-algebra in Z and $\mathcal{P}_t$ in $\tilde{\Omega}$, see [13].*

Theorem 14. *Specify an arbitrary sequence of positive numbers $\varepsilon_k \to 0$ as $k \to \infty$. Let $\boldsymbol{B}$ be an upper semi-continuous set-valued mapping with compact convex values sending $[0,T]\times\tilde{\Omega}$ to a finite-dimensional Euclidean vector space Y and satisfying Condition 12. Then there exists a sequence of continuous single-valued mappings $B_k:[0,T]\times\tilde{\Omega} \to Y$ with the following properties:*

(i) each B_k satisfies Condit*ion 12;*

(ii) the sequence B_k point-wise converges to a selector of $\boldsymbol{B}$ that is measurable with respect to Borel σ-algebra in Y and the product σ-algebra of Borel one on $[0,T]$ and $\tilde{\mathcal{F}}$ on $\tilde{\Omega}$;

(iii) at each $(t,x(\cdot)) \in [0,T]\times\tilde{\Omega}$ the inequality $\|B_k(t,x(\cdot))\| \leq \|\boldsymbol{B}(t,x(\cdot))\|$ holds for all k;

(iv) if $\boldsymbol{B}$ takes values in a closed convex set $\Xi \subset Y$, the values of all B_k belong to Ξ.

Proof. In this proof we combine and modify the ideas used in the proofs of [10] by Gel'man and Theorem 11 above.

For $t \in [0,T]$ introduce the mapping $f_t:\tilde{\Omega} \to \tilde{\Omega}$ by the formula

$$f_t x(\cdot) = \begin{cases} x(s) \text{ if } 0 \leq s \leq t \\ x(t) \text{ if } t \leq s \leq T. \end{cases} \quad (3)$$

Obviously $f_t x(\cdot)$ is continuous jointly in $t \in [0,T]$ and $x(\cdot) \in \tilde{\Omega}$. Since **B** satisfies Condition 12, $\boldsymbol{B}(t,x(\cdot)) = \boldsymbol{B}(t,f_t x(\cdot))$ for each $x(\cdot) \in \tilde{\Omega}$ and $t \in [0,T]$.

Specify an element ε_k from the sequence. Since $\boldsymbol{B}$ is upper semi-continuous, for every $(t,x(\cdot)) \in [0,T]\times\tilde{\Omega}$ there exists $\delta_k(t,x) > 0$ such that for every $(t^*,x^*(\cdot))$ from the $\delta_k(t,x)$ neighbourhood of $(t,x(\cdot))$ the set $\boldsymbol{B}(t^*,x^*(\cdot))$ is contained in the $\frac{\varepsilon_k}{2}$-neighbourhood of the set $\boldsymbol{B}(t,x(\cdot))$. Without loss of generality we can suppose $0 < \delta_k(t,x) < \varepsilon_k$ for every $(t,x(\cdot))$. Consider the $\frac{\delta_k(t,x)}{4}$-neighbourhood of $(t,x(\cdot))$ in $[0,T]\times\tilde{\Omega}$ and construct the open covering of $[0,T]\times\tilde{\Omega}$ by such neighbourhoods for all $(t,x(\cdot))$. Since $[0,T]\times\tilde{\Omega}$ is paracompact, there exists a locally finite refinement $\{V_j^k\}$ of this covering. Without loss of generality we can consider each V_j^k as an $\eta_k(t_j^k,x_j^k)$-neighbourhood of a certain $(t_j^k,x_j^k(\cdot))$ where by construction the radius $\eta_k(t_j,x_j) \leq \frac{\delta_k(t_j,x_j)}{4}$.

Consider a continuous partition of unity $\{\varphi_j^k\}$ adapted to $\{V_j^k\}$ and introduce the set-valued mapping $\Phi_k(t,x(\cdot)) = \sum_j \varphi_j^k(t,x(\cdot))\overline{co}\boldsymbol{B}(V_j^k)$ where $\overline{co}$ denotes the convex closure. Since $\boldsymbol{B}(t,x(\cdot))$ is upper semi-continuous and has compact values, without loss of generality we can suppose $\delta_k(t,x)$ to be such that the images $\boldsymbol{B}(V_j^k)$ are bounded in Y and so the sets $\overline{co}\boldsymbol{B}(V_j^k)$ are compact. Denote by $\overline{\mathbf{\Phi}}_k(t,x(\cdot))$ the closure of $\mathbf{\Phi}_k(t,x(\cdot))$. Then one can easily see that $\overline{\mathbf{\Phi}}_k:[0,T]\times\tilde{\Omega}\to Y$ is a Hausdorff continuous set-valued mapping with compact convex values.

Introduce $\mathbf{\Psi}_k:[0,T]\times\tilde{\Omega}\to Y$ by formula $\mathbf{\Psi}_k(t,x(\cdot))=\mathbf{\Phi}_k(t,f_tx(\cdot))$ and consider the set-valued mapping $\overline{\mathbf{\Psi}}_k(t,x(\cdot))$. Since f_t is continuous, every $\overline{\mathbf{\Psi}}_k$ is a Hausdorff continuous set-valued mapping with compact convex values and by construction it satisfies Condition 12.

The couple $(t,f_tx(\cdot))$ belongs to a finite collection of neighbourhoods $V^k_{j_i}$ with centers at $(t^k_{j_i},x^k_{j_i}(\cdot))$, $i=1,\dots,n$ and so by construction $\boldsymbol{B}(t,x(\cdot))=\boldsymbol{B}(t,f_tx(\cdot))\subset\boldsymbol{B}(V^k_{j_i})$ for each i. Hence $\boldsymbol{B}(t,x(\cdot))=\boldsymbol{B}(t,f_tx(\cdot))\subset\mathbf{\Psi}_k(t,x(\cdot))$ for every couple $(t,x(\cdot))$.

Let l be the number from the collection of indices j_i as above such that $\eta_k(t^k_l,x^k_l)$ takes the greatest value among $\eta_k(t^k_{j_i},x^k_{j_i})$. Then all $(t^k_{j_i},x^k_{j_i}(\cdot))$ are contained in the $2\eta_k(t^k_l,x^k_l)$-neighbourhood of $(t^k_l,x^k_l(\cdot))$ and so every $V^k_{j_i}$ is contained in $3\eta_k(t^k_l,x^k_l)$-neighbourhood *of* $(t^k_l,x^k_l(\cdot))$ that is contained in $\delta_k(t^k_l,x^k_l(\cdot))$-neighbourhood of (t^k_l,x^k_l) by construction. Then, also by construction, $\mathbf{\Psi}_k(t,x(\cdot))$ belongs to the $\dfrac{\varepsilon_k}{2}$-neighbourhood of $\boldsymbol{B}(t^k_l,x^k_l(\cdot))$. Since both $\mathbf{\Psi}_k(t,x(\cdot))$ and $\boldsymbol{B}(t^k_l,x^k_l(\cdot))$ are convex, this means that $\overline{\mathbf{\Psi}}_k(t,x(\cdot))$ also belongs to the $\dfrac{\varepsilon_k}{2}$-neighbourhood of $\boldsymbol{B}(t^k_l,x^k_l(\cdot))$. Notice that this is true for each k.

Since $\boldsymbol{B}(t,x(\cdot))\subset\mathbf{\Psi}_k(t,x(\cdot))\subset\overline{\mathbf{\Psi}}_k(t,x(\cdot))$, for the Hausdorff submetric $\bar{H}$ we have

$$\bar{H}(\boldsymbol{B}(t,x(\cdot)),\overline{\mathbf{\Psi}}_k(t,x(\cdot)))=0.$$

Hence for the Hausdorff metric H we obtain that

$$H(\overline{\mathbf{\Psi}}_k(t,x(\cdot)),\boldsymbol{B}(t,x(\cdot)))=\bar{H}(\overline{\mathbf{\Psi}}_k(t,x(\cdot)),\boldsymbol{B}(t,x(\cdot))).$$

Since $\varepsilon_k\to 0$, for $(t,x(\cdot))$ there exists an integer $\theta=\theta(t,x(\cdot))>0$ such that $\varepsilon_{k+\theta}<\delta_k(t,x(\cdot))$. Without loss of generality we can suppose that $\theta\geq 1$.

Thus $\mathbf{B}(t^{k+\theta}_l,x^{k+\theta}_l(\cdot))$ belongs to the $\dfrac{\varepsilon_k}{2}$-neighbourhood of $\mathbf{B}(t,x(\cdot))$ and so

$$\bar{H}(\boldsymbol{B}(t^{k+\theta}_l,x^{k+\theta}_l(\cdot)),\boldsymbol{B}(t,x(\cdot)))<\frac{\varepsilon_k}{2}.$$

Since $\overline{\mathbf{\Psi}}_{k+\theta}(t,x(\cdot))$ belongs to the $\dfrac{\varepsilon_{k+\theta}}{2}$-neighbourhood of $\boldsymbol{B}(t^{k+\theta}_l,x^{k+\theta}_l(\cdot))$ (see above), we obtain that

$$\bar{H}(\overline{\mathbf{\Psi}}_{k+\theta}(t,x(\cdot)),\boldsymbol{B}(t^{k+\theta}_l,x^{k+\theta)}_l(\cdot)))<\frac{\varepsilon_{k+\theta}}{2}.$$

Thus

$$H(\overline{\mathbf{\Psi}}_{k+\theta}(t,x(\cdot)),\boldsymbol{B}(t,x(\cdot)))=$$
$$=\bar{H}(\overline{\mathbf{\Psi}}_{k+\theta}(t,x(\cdot)),\boldsymbol{B}(t,x(\cdot)))\leq$$
$$\leq\bar{H}(\overline{\mathbf{\Psi}}_{k+\theta}(t,x(\cdot)),\boldsymbol{B}(t^{k+\theta}_l,x^{k+\theta}_l(\cdot)))+$$
$$+\bar{H}(\boldsymbol{B}(t^{k+\theta}_l,x^{k+\theta}_l(\cdot)),\boldsymbol{B}(t,x(\cdot)))<\frac{\varepsilon_{k+\theta}}{2}+\frac{\varepsilon_k}{2}<\varepsilon_k.$$

So, at each $(t,x(\cdot))$ we have that $H(\overline{\mathbf{\Psi}}_k(t,x(\cdot)),\boldsymbol{B}(t,x(\cdot)))\to 0$ as $k\to\infty$ and $\boldsymbol{B}(t,x(\cdot))\subset\mathbf{\Psi}_k(t,x(\cdot))$ for all k.

Consider the minimal selector $B_k(t,x(\cdot))$ of $\overline{\mathbf{\Psi}}_k(t,x(\cdot))$, i.e., $B_k(t,x(\cdot))$ is the closest to origin point in $\overline{\mathbf{\Psi}}_i(t,x(\cdot))$. We refer the reader to [11] for complete description of minimal selectors. In particular, it is shown there that minimal selectors in our situation are continuous. By construction all B_k satisfy Condition 12.

By construction the minimal selectors $B_k(t,x(\cdot))$ of $\overline{\mathbf{\Psi}}_k(t,x(\cdot))$ point-wise converge to the minimal selector $B(t,x(\cdot))$ of $\boldsymbol{B}(t,x(\cdot))$ as $k\to\infty$ since at any $(t,x(\cdot))$ we have that $H(\overline{\mathbf{\Psi}}_k(t,x(\cdot)),\boldsymbol{B}(t,x(\cdot)))\to 0$ as $k\to\infty$ and $\boldsymbol{B}(t,x(\cdot))\subset\mathbf{\Psi}_k(t,x(\cdot))$ for all k (see above). It is a well-known fact that the point-wise limit B of the sequence of continuous mappings B_k is measurable with respect to Borel σ-algebras in Y and in $[0,T]\times\tilde{\Omega}$ (see [14]). The latter coincides with the product σ-algebra of Borel one on $[0,T]$ and $\tilde{\mathcal{F}}$ on $\tilde{\Omega}$ (see [12]). Properties *(iii)* and *(iv)* immediately follow from the construction. ■

Remark 15. *Unlike* $\bar{\mathbf{\Psi}}_k(t,x(\cdot))$, *the set-valued mapping* $\bar{\mathbf{\Phi}}_k(t,x(\cdot))$ *may not satisfy Condition 12 since two different curves* $x_1(\cdot)$ *and* $x_2(\cdot)$ *coinciding on* $[0,t]$, *may have different neighbourhoods* V^k_j, *to which they belong, and so the values* $\bar{\mathbf{\Phi}}_k(t,x_1(\cdot))$ *and* $\bar{\mathbf{\Phi}}_k(t,x_2(\cdot))$ *may be different. On the other hand, it follows from [10] that* $\bar{\mathbf{\Phi}}_k$ *is an* ε_k*-approximation of* $\boldsymbol{B}$ *while it is not true for* $\bar{\mathbf{\Psi}}_k$.

4. SPECIAL ε-APPROXIMATIONS FOR UPPER SEMI-CONTINUOUS MAPPINGS WITH ASPHERIC CLOSED VALUES

We start this section from the exact definition of a set-valued mapping with aspheric values (see [6, 7, 8]).

Below in this section we consider a set-valued upper semi-continuous mapping $F:X\multimap E$ from a n-dimensional polyhedron X lying in some Euclidean space, to a certain Euclidean space E.

The assumption that X is a polyhedron does not lose generality. In particular, we can consider $F : X \multimap X$ and $F : E \multimap E$ since an Euclidean space can be presented as a polyhedron.

By $O(A, r)$ we denote an r-neighbourhood of the set A and by $d(A)$ — the diameter of A.

Recall that from the definition of upper semicontinuity of F it follows that for every $\varepsilon > 0$ and $\beta > 0$ there exists $\alpha(\varepsilon, \beta)$ such that in a β-neighbourhood $O(T, \beta)$ of an arbitrary set T with diameter less than α there exists a point x_0, called *satellite* of T, for which $O(F(x_0), \varepsilon) \supset F(T)$.

Definition 16. *The mapping $F : X \multimap X$ is called aspheric in a dimension k if in every neighbourhood $O(F(x), \varepsilon)$ of each value $F(x)$ there exists a neighbourhood $Q(x, \varepsilon, k)$ containing $\delta = \delta(\varepsilon)$ neighbourhood of $F(x)$ (δ does not depend on x), such that $\pi_k(Q) = 0$ where $\pi_k(Q)$ is the k-th homotopy group of Q.*

Everywhere below we suppose that F is aspheric in dimensions $k = 0, 1, \dots, n-1$. Recall that $\pi_0(Q) = 0$ means that Q is arcwise connected. For such upper semicontinuous mappings with closed values we describe the construction of ε-approximations following [6].

Let μ be a real number such that $O(F(x), \mu)$ belongs to an aspheric in dimension $n-1$ neighbourhood of $F(x)$, μ does not depend on x. Construct a sequence

$$\begin{gathered} \mu > \varepsilon_{2n+1} > \varepsilon_{2n} > \delta(\varepsilon_{2n}) > \\ > \varepsilon_{2n-1} > \dots > \varepsilon_2 > \delta(\varepsilon_2) > \varepsilon_1, \end{gathered} \tag{4}$$

where $\delta(\varepsilon_i)$ is a number that determines $\delta(\varepsilon_i)$-neighbourhood of the value $F(x)$ contained in $\bigcap_{k=0}^{n} Q(x, \varepsilon_i, k)$. Then construct a sequence $\{\beta_i\}_1^{n+1}$ and a number α_0 such that

$$0 < \beta_k < \frac{1}{4}\beta_{k+1}; \beta_k + \alpha_0 < \alpha(\varepsilon_{2k+1} - \varepsilon_{2k}, \beta_{k+1}), \tag{5}$$

where $\alpha(\varepsilon, \beta)$ is introduced above in this section. Such sequences evidently can be constructed starting from greater indices.

Now construct a triangulation of X whose mesh is such that the diameter of every simplex is less than $d < \min(\alpha_0, \alpha(\varepsilon_1, \beta_1))$. To every 0-dimensional simplex T_i^0 we assign a point $f(T_i^0) \in F(T_i^0)$. For every 1-dimensional simplex T_i^1 we get $d(T_i^1) < \alpha(\varepsilon_1, \beta_1)$. So, there exists a satellite x_i^1 such that $x_i^1 \in O(T_i^1, \beta_1)$ and $F(T_i^1) \subset O(F(x_i^1), \varepsilon_1)$. Hence the following inclusions take place:

$$f(T_{i_1}^0) \cup f(T_{i_2}^0) \subset F(T_i^1) \subset O(F(x_i^1), \varepsilon_1) \tag{6}$$

and

$$\begin{gathered} O(F(x_i^1), \varepsilon_1) \subset O(F(x_i^1), \delta(\varepsilon_2)) \subset \\ \subset Q(x_i^1, \varepsilon_2, 0) \subset O(F(x_i^1), \varepsilon_2), \end{gathered} \tag{7}$$

where $T_{i_1}^0$ and $T_{i_2}^0$ are sides of T_i^1. Since Q is aspheric in dimension 0, f can be extended to T_i^1 as a continuous mapping and

$$f(T_i^1) \subset Q(x_i^1, \varepsilon_2, 0) \subset O(F(x_i^1), \varepsilon_2). \tag{8}$$

Let T_i^2 be a 2-dimensional simplex with 1-dimensional sides $T_{i_1}^1$, $T_{i_2}^1$ and $T_{i_3}^1$. Let $x_{i_1}^1$, $x_{i_2}^1$ and $x_{i_3}^1$ be the satellites corresponding to those sides. They form the set $\tilde{T}_i^1$, for which

$$d(\tilde{T}_i^1) < 2\beta_1 + \alpha_0 < \alpha(\varepsilon_3 - \varepsilon_2, \beta_2).$$

There exists a satellite x_i^2 of , such that

$$x_i^2 \in O(\tilde{T}_i^1, \beta_2) \text{ and } F(\tilde{T}_i^1) \subset O(F(x_i^2), \varepsilon_2).$$

Taking into account (7) and (8) we derive

$$\bigcup_{j=1,2,3} f(T_{i_j}^1) \subset O(F(\tilde{T}_i^1), \varepsilon_2) \subset O(F(x_i^2), \varepsilon_2). \tag{9}$$

By (4) we have the inclusions

$$\begin{gathered} O(F(x_i^2), \varepsilon_3) \subset O(F(x_i^2), \delta(\varepsilon_4)) \subset \\ \subset Q(x_i^2, \varepsilon_4, 1) \subset O(F(x_i^2, \varepsilon_4). \end{gathered} \tag{10}$$

Since $\pi_2 Q(x_i^2, \varepsilon_4, 1) = 0$, we can extend f from the boundary of simplex T_i^2 onto the entire simplex as a continuous mapping. In addition we obtain that

$$f(T_i^2) \subset Q(x_i^2, \varepsilon_4, 1) \subset O(F(x_i^2), \varepsilon_4). \tag{11}$$

And so on. On the last step we extend f from the $(n-1)$-skeleton of X onto entire X as a continuous mapping. By construction, the graph of f lies in an ε_{2n+1}-neighbourhood on the graph of F.

Theorem 17. *For F as above there exists a sequence $f^{(k)}$ of continuous ε_{2n+1}^k-approximations of the type constructed above, $\varepsilon_{2n+1}^k \to 0$ as $k \to \infty$, such that for every point x of some countable everywhere dense subset $\Xi \subset X$ there exists an integer K such that for every $k > K$ the inclusion $f^{(k)}(x) \in F(x)$ holds and $f^{(k+l)}(x) = f^{(k)}(x)$ for every integer $l > 0$.*

Proof. By construction, for every x from the 0-dimensional skeleton of X for f constructed above, we have $f(x) \in F(x)$. Now construct a sequence of barycentric subdivisions of X. Denote by X_0^k the 0-dimensional skeleton of k-th subdivision. On every step $k+1$ for $x \in X_0^k$ keep the value $f^{(k+1)}(x) = f^{(k)}(x)$ and introduce an arbitrary value $f^{(k+1)}(x) \in F(x)$ for $x \in X_0^{(k+1)} \setminus X_0^{(k)}$. Then construct continuous $f^{(k+1)}$ on the entire X in the same manner as above. The limit Ξ of $X_0^{(k)}$ as $k \to \infty$ is the set we are looking for. ■

Corollary 18. *In notation of Theorem 17 the sequence $f^{(k)}$ on Ξ point-wise converges to a selector f of F so that for every $x \in \Xi$ the values $f^{(k)}(x)$ stabilize starting from a certain integer $K(x)$. From the point-wise convergence if follows that f is Borel measurable on Ξ.*

Remark 19. *In [6,8] it is shown that under additional assumption that F is weakly aspheric in dimension n (i.e., there exists a certain $\varepsilon > 0$ such that $\pi_n Q = 0$ as in Definition 16) the homotopy class of approximations $f^{(k)}$ stabilizes for k large enough. This allows one to introduce the homotopic characteristics for F as those for stabilized $f^{(k)}$.*

REFERENCES

1. *Azarina, S.V.* Solvability of Langevin differential inclusions with set-valued diffusion terms on Riemannian manifolds / S. V. Azarina, Yu. E. Gliklikh, A. V. Obukhovskii // Applicable Analysis. — 2007. — Vol. 86, № 9. — P. 1105—1116

2. *Azarina, S.V., Gliklikh, Yu.E.* Stochastic differential equations and inclusions with mean derivatives relative to the past / S. V. Azarina, Yu. E. Gliklikh // International Journal of Difference Equations (in print)

3. *Borisovich Yu.G.* Introduction to the theory of set-valued mappings and differential inclusions / Yu. G. Borisovich, B. D. Gel'man, A. D. Myshkis, V. V. Obukhovskiĭ. — Moscow: KomKniga, 2005. — 216 p. (in Russian)

4. *Kamenskiĭ M.* Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces / M. Kamenskiĭ, V. Obukhovskiĭ, P. Zecca. — Berlin-New York: Walter de Gruyter, 2001. — 231 p.

5. *Deimling K.* Multivalued differential equations / K. Deimling. — Berlin-New York: Walter de Gruyter, 1992. — 257 p.

6. *Borisovich Yu.G.* On Lefschetz number for a certain class of set-valued maps / Yu. G. Borisovich, Yu. E. Gliklikh // Trokhimchuk Yu. Yu. (ed.) 7-th Summer Mathematical School. — Kiev, 1970. — P. 283—294. (Russian)

7. *Myshkis A.D.* Generalization of the theorem on the stationary point of the dynamical system inside a closed trajectory / A. D. Myshkis // Mat. sbornik. — 1954. — Vol. 34, № 3. — P. 525—540

8. *Gliklikh Yu.E.* Fixed points of multivalued mappings with nonconvex images and the rotation of multivalued vector fields / Yu. E. Gliklikh // Borisovich Yu. G. (ed.) Sbornik Trudov Aspirantov Matematicheskogo Fakul'teta. — Voronezh: Voronezh University, 1972. — P. 30—38 (Russian)

9. *Kryszewski W.* Homotopy properties of set-valued mappings / W. Kryszewski // Torun: Torun University, 1997. — 243 p.

10. *Gel'man B.D.* Continuous approximations of set-valued mappings and fixed points / B. D. Gel'man // Mathematical Notices. — 2005. — Vol. 78, № 2. — P. 212—222

11. *Aubin J.-P.* Differential Inclusions. Set-valued maps and viability theory / J.-P. Aubin, A. Cellina. — Berlin et al.: Springer-Verlag, 1984. — 342 p.

12. *Shiryaev A.N.* Probability / A. N. Shiryaev // New York Berlin Heidelberg: Springer-Verlag, 1984. — 640 p.

13. *Gihman I.I.* Theory of Stochastic Processes / I. I. Gihman, A. V. Skorohod. — New York: Springer-Verlag, 1979. — Vol. 3. — 495 p.

14. *Parthasarathy K.R.* Introduction to Probability and Measure / K. R. Parthasarathy. New York: Springer-Verlag, 1978. — 344 p.

Гликлих Юрий Евгеньевич — доктор физико-математических наук, профессор кафедры алгебры и топологических методов анализа Воронежского государственного университета, тел. +7(4732)208641, e-mail: yeg@math.vsu.ru

Gliklikh Yuri E. — ScD, Full Professor of Voronezh State University, tel. +7(4732)208641, e-mail: yeg@math.vsu.ru