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ON RELATIONS BETWEEN INFINITESIMAL GENERATORS AND MEAN DERIVATIVES OF STOCHASTIC PROCESSES ON MANIFOLDS*

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Аннотация. Описываются соотношения между инфинитезимальными генераторами слева и справа с одной стороны и производными в среднем слева и справа с другой стороны для случайных процессов на многообразиях. Эти соотношения формулируются в терминах порожденного аффинной связностью отображения из расслоения векторов второго порядка в обычное (т.е. первого порядка) касательное расслоение. Также показано, что так называемая квадратичная производная в среднем может быть получена из генератора с помощью другого морфизма соответствующих расслоений.

Ключевые слова: Случайные процессы; инфинитезимальные генераторы; производные в среднем; связности на многообразиях; касательные векторы второго порядка

Abstract. We describe the relation between the forward (backward) infinitesimal generator on the one hand and forward (backward, respectively) mean derivative on the other hand for a stochastic process on manifold. The relation is formulated in terms of a mapping from the second order tangent bundle to the first order one, generated by a given connection. It is also shown that the so-called quadratic mean derivative can be obtained from the generator by another morphism between the corresponding bundles.

Key words: Stochastic processes; infinitesimal generators; mean derivatives; connections on manifolds; second order tangent vectors.

There are two types of differential operators associated to a stochastic process: the infinitesimal generators (forward and backward) and mean derivatives (forward, backward and quadratic). Recall that the generators are determined invariantly as the so-called second order tangent vectors, quadratic mean derivatives are invariant as well and take values in $(2,0)$-tensors while the forward and backward mean derivatives on a manifold are well defined for a given connection and then take values in first order vectors.

In this paper we show that given a connection, one can construct forward (backward) mean derivative from forward (backward, respectively) generator by application of a natural mapping from second order tangent bundle to the first order one, generated by the connection. The quadratic mean derivative can be obtained from the generator by another special operator between the corresponding bundles that is independent of the choice of connection.

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Let M be a smooth manifold of dimension n.

Let $\mathcal{U}_\alpha$ be a chart on M. Denote by $q^1,\dots,q^n$ the local coordinates in U_α and by $\frac{\partial}{\partial q^1},\dots,\frac{\partial}{\partial q^n}$ the corresponding basis vectors in tangent spaces to $\mathcal{U}_\alpha$ (we do not distinguish in notations the tangent vectors and the corresponding first order differential operators). Consider a differential operator in $\mathcal{U}_\alpha$ of order no greater than 2 without constant term of the form

$$B(t,m) = b^i \frac{\partial}{\partial q^i} + \beta^{ij} \frac{\partial^2}{\partial q^i \partial q^j}, \qquad (1)$$

where the coefficients β^{ij} form a symmetric matrix (β^{ij}).

Definition 1. *A second order tangent vector to a manifold M at a point $m \in M$ is a differential operator of the order no greater that 2 without constant term as in (1) that has symmetric matrix of coefficients at second order derivatives. The linear space of second order tangent vectors at $m \in M$ is called the second order tangent space and is denoted by $\tau_m M$. The second order tangent bundle is denoted by $\tau(M)$.*

Notice that at every $m \in M$ the first order tangent space T_mM is a subspace in $\tau_m M$ — the first order vectors have zero matrix (β^{ij}). On the other hand, if that matrix is not zero, the column (b^i) is not a first order tangent vector since it has another rule for transformations: under coordinate changes the pure second order term transforms into the one with additional first order term. Nevertheless anyhow the field of matrices (β^{ij}) is a symmetric $(2,0)$-tensor field.

At every $m \in M$ there is a canonical isomorphisms between the space $T_mM \odot T_mM$ (where T_mM is the first order, i.e., ordinary tangent space, $\odot$ denotes the symmetric tensor product) and the quotient space $\tau_m M/T_mM$, and so between $TM \odot TM$ and $\tau M/TM$. Denote by

$$\mathcal{Q} : \tau M \to TM \odot TM$$

the field of linear projectors

$$\mathcal{Q}_m : \tau_m M \to T_mM \odot T_mM$$

determined by the above factorization. Note that the sections of $TM \odot TM$ are symmetric $(2,0)$-tensor fields and that by construction

$$\mathcal{Q}B(t,m) = = \mathcal{Q}\left(b^i \frac{\partial}{\partial q^i} + \beta^{ij} \frac{\partial^2}{\partial q^i \partial q^j}\right) = (\beta^{ij}). \tag{2}$$

Specify an arbitrary connection $\mathcal{H}$ on M with Christoffel symbols of second kind Γ^k_{ij}. It determines the morphism $\mathcal{H} : \tau M \to TM$ (i.e., the smooth field of linear operators $\mathcal{H}_m : \tau_m M \to T_mM$) of the form

$$\mathcal{H}B(t,m) = b^k \frac{\partial}{\partial q^k} + \Gamma^k_{ij}\beta^{ij} \frac{\partial}{\partial q^k} \tag{3}$$

(see [1—3]). One can easily verify that the right-hand side of (3) transforms like a first order tangent vector under the changes of coordinates.

Let $\mathbb{R}^N$ be a certain N-dimensional linear space. Denote by $L(\mathbb{R}^N, \mathbb{R}^n)$ the space of linear operators sending $\mathbb{R}^N$ to $\mathbb{R}^n$ where n is the dimension of M as above.

Definition 2. (cf. [4—7]) *The Itô bundle $I(M)$ over M is the bundle such that over a chart $\mathcal{U}_\alpha$ on M it is presented as direct product $\mathcal{U}_\alpha \times (\mathbb{R}^n \times L(\mathbb{R}^N, \mathbb{R}^n))$ (i.e., $\mathbb{R}^n \times L(\mathbb{R}^N, \mathbb{R}^n)$ is the standard fiber of $I(M)$) and under the change of coordinates $\varphi_{\beta\alpha}$ from $\mathcal{U}_\alpha$ to another chart $\mathcal{U}_\beta$ it transforms according to the rule*

$$(m,(X,A)) \mapsto \mapsto \left(\varphi_{\beta\alpha} m, \left(\varphi'_{\beta\alpha} X + \frac{1}{2} tr\varphi''_{\beta\alpha}(A,A), \varphi'_{\beta\alpha} A\right)\right). \tag{4}$$

where $\varphi'_{\beta\alpha}$ and $\varphi''_{\beta\alpha}$ are the first and the second derivatives of $\varphi_{\beta\alpha}$, respectively.

The (generally speaking non-autonomous) sections of $I(M)$ are called Itô equations.

The backward Itô bundle $I_*(M)$ over M is the bundle such that over a chart $\mathcal{U}_\alpha$ on M it is presented as direct product $\mathcal{U}_\alpha \times (\mathbb{R}^n \times L(\mathbb{R}^N, \mathbb{R}^n))$ (i.e., $\mathbb{R}^n \times L(\mathbb{R}^N, \mathbb{R}^n)$ is the standard fiber of $I(M)$) and under the change of coordinates $\varphi_{\beta\alpha}$ from $\mathcal{U}_\alpha$ to another chart $\mathcal{U}_\beta$ it transforms according to the rule

$$(m,(X,A)) \mapsto \mapsto \left(\varphi_{\beta\alpha} m, \left(\varphi'_{\beta\alpha} X - \frac{1}{2} tr\varphi''_{\beta\alpha}(A,A), \varphi'_{\beta\alpha} A\right)\right). \tag{5}$$

where $\varphi'_{\beta\alpha}$ and $\varphi''_{\beta\alpha}$ are the first and the second derivatives of $\varphi_{\beta\alpha}$, respectively.

The (generally speaking non-autonomous) sections of $I_*(M)$ are called backward Itô equations.

In the chart U_α an Itô equation is presented as a couple $(\hat{a}(t,m), A(t,m))$ where $A(t,m)$ is a linear operator from $\mathbb{R}^N$ to the tangent space T_mM. We keep the notation of Itô equation as the couple everywhere in spite of the fact that it is well-posed only in a chart.

Note that $\hat{a}(t,m)$ is not a tangent vector to M. Nevertheless with respect to the given trivialization of $I(M)$ over $\mathcal{U}_\alpha$ we can represent $\hat{a}(t,m)$ as a column of coordinates $(\hat{a}^i)$. Denote by $(A^j_i)(t,m)$ the matrix of $A(t,m)$ with respect to the same trivialization.

Introduce a Wiener process $w(t)$ in $\mathbb{R}^N$ given on a certain probability space. Then we can consider an Itô stochastic differential equation

$$d\xi(t) = \hat{a}(t,\xi(t))dt + A(t,\xi(t))dw(t) \tag{6}$$

in $\mathcal{U}_\alpha$. Taking into account the Itô formula, one can easily see that a solution $\xi(t)$ of (6) is presented in U_β in the form

$$d\varphi_{\beta\alpha}\xi(t) = \varphi'_{\beta\alpha}\hat{a}dt + \frac{1}{2} tr\varphi''_{\beta\alpha}(A,A)dt + +\varphi'_{\beta\alpha}A\,dw(t). \tag{7}$$

Thus from (4) and (7) it obviously follows that a solution $\xi(t)$ is well-posed on the entire manifold (see details in [4—7]). We shall call it a solution of $(\hat{a}(t,m), A(t,m))$ on M.

Take a relatively compact chart $\mathcal{U}_\alpha$. Denote by θ_{U_α} the random time of the first entrance to the boundary of $\mathcal{U}_\alpha$ by a stochastic process $\xi(\cdot)$ with $\xi(t) \in \mathcal{U}_\alpha$ and by $\theta^*_{\mathcal{U}_\alpha}$ its last exit time from the boundary.

We define the infinitesimal generator of $\xi(\cdot)$ as the field of differential operators $G(t,m)$ that acts on a smooth enough real valued function $f: M \to \mathbb{R}$ by the formula

$$G(t,m)f = \\ = \lim_{\Delta t \to +0} E\left(\frac{f(\xi((t+\Delta t) \wedge \theta_{U_\alpha}) - f(\xi(t))}{\Delta t} \mid \xi(t) = m \right)$$

and the backward infinitesimal generator $G_*(t,m)$ as

$$G_*(t,m)f = \\ = \lim_{\Delta t \to +0} E\left(\frac{f(\xi(t)) - f(\xi((t-\Delta t) \vee \overset{*}{\theta}_{U_\alpha})}{\Delta t} \mid \xi(t) = m \right).$$

By routine calculations one can easily see that in U_α the generator L of a solution $\xi(t)$ of Itô equation $(\hat{a}(t,m), A(t,m))$ takes the form

$$L(t,x) = \hat{a}^i \frac{\partial}{\partial q^i} + \frac{1}{2}\alpha^{ij} \frac{\partial^2}{\partial q^i \partial q^j} \qquad (8)$$

where the matrix $(\alpha^{ij})(t,m)$ is the matrix product $(A^k_l)(A^k_l)^*$ and $(A^k_l)^*$ is the transposed matrix to (A^k_l), i.e., the matrix of conjugate operator. It is also easy to see that operator L from (8) is a second order tangent vector field on M.

A couple $(a(t,m), A(t,m))$ where $a(t,m)$ is a (first order) vector field and $A(t,m)$ is a field of linear operators from $\mathbb{R}^N$ to $T_m M$, is called an Itô vector field. Notice that the transformation rule under changes of coordinates on M for an Itô vector field coincides with that for ordinary tangent vectors.

Of course Itô equations and Itô vector fields cannot coincide since they have different transformation rules under changes of coordinates. They may coincide for given trivializations over a certain chart but this coincidence will become failed in other trivializations (over the other charts).

Specify a connection $\mathcal{H}$ on M.

Definition 3. *We say that an Itô vector field* $(a(t,m), A(t,m))$ *and an Itô equation* $(\hat{a}(t,m), \hat{A}(t,m))$ *canonically correspond to each other with respect to connection* $\mathcal{H}$ *if at every* $m \in M$ *and for all* t *from the domain,* $a(t,m) = \hat{a}(t,m)$ *and* $A(t,m) = \hat{A}(t,m)$ *with respect to trivialization of normal chart of* $\mathcal{H}$ *at* m.

From definition of $A(t,m)$ and $\hat{A}(t,m)$ it follows that if $A(t,m) = \hat{A}(t,m)$ in a certain chart, the equality remains true in every chart.

Theorem 4. *An Itô vector field* $(a(t,m), A(t,m))$ *and an Itô equation* $(\hat{a}(t,m), \hat{A}(t,m))$ *canonically correspond to each other with respect to connection* $\mathcal{H}$ *if and only if* $A(t,m) = \hat{A}(t,m)$ *and in every chart for every* $k = 1,\dots,n$ *the equality* $\hat{a}^k(t,m) = a^k(t,m) - \frac{1}{2}\Gamma^k_{ij}\alpha^{ij}$ *holds where* $(\alpha^{ij}) = (A^i_k)(A^k_j)^*$ *and* Γ^k_{ij} *are Christoffel symbols of second kind for* $\mathcal{H}$ *in the chart.*

Theorem 1 is a reformulation of lemma 9.8 from [5] (see also lemma 11.21 in [7]).

The object with coordinates $\Gamma^k_{ij}\alpha^{ij}$ is $tr\Gamma_m(A,A)$ where $\Gamma_m(\cdot,\cdot)$ is the so called local connector of $\mathcal{H}$ in the chart (we shall not describe this notion here, see details, e.g., in [5—7]). Thus for a given connection equation (6) can be presented in equivalent form

$$d\xi(t) = a(t,\xi(t))dt - \\ - \frac{1}{2} tr\Gamma_{\xi(t)}(A,A)dt + A(t,\xi(t))dw(t) \qquad (9)$$

where (a, A) is the Itô vector field canonically corresponding to $(\hat{a}, A)$ with respect to $\mathcal{H}$. Equation (9) is known as Itô equation in Backsendale's form. It is a presentation in local chart of invariant equations known as Itô equation in Belopolskaya—Daletskii form (see [4—7]).

Theorem 2. *Let the Itô vector field* $(a(t,m), A(t,m))$ *canonically correspond to an Itô equation* $(\hat{a}(t,m), A(t,m))$ *with respect to connection* $\mathcal{H}$ *and let* L *be the generator of solutions of* $(\hat{a}(t,m), A(t,m))$. *Then* $a(t,m) = \mathcal{H}L$ *where* $\mathcal{H}$ *is given by* (3).

Proof. Indeed, by (10) in a chart $\mathcal{U}_\alpha$ we have $L(t,x) = \hat{a}^i \frac{\partial}{\partial q^i} + \frac{1}{2}\alpha^{ij}\frac{\partial^2}{\partial q^i \partial q^j}$. Then by (3)

$$\mathcal{H}L = \hat{a}^k \frac{\partial}{\partial q^k} + \frac{1}{2}\Gamma^k_{ij}\alpha^{ij}\frac{\partial}{\partial q^k}.$$

But by Theorem 1 $\hat{a}^k(t,m) = a^k(t,m) - \frac{1}{2}\Gamma^k_{ij}\alpha^{ij}$. This completes the proof. ■

Let $\xi(t)$ be a stochastic process with values in M given on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Denote by E^ξ_t the conditional expectation with respect to the σ-subalgebra $\mathcal{N}^\xi_t$ generated by preimages of Borel sets in M relative to the mapping $\xi(t): \Omega \to M$ (the "present" of process $\xi(t)$). In [8—13] the notion of forward mean derivative, backward mean derivative and quadratic mean derivative were introduced as follows.

Take a relatively compact normal chart $\mathcal{U}_m$ of connection $\mathcal{H}$ at a specified point $m \in M$. Denote by θ_{U_m} the random time of the first entrance of

$\xi(\cdot)$ to the boundary of $\mathcal{U}_m$ with $\xi(t) \in \mathcal{U}_m$. Introduce the notation

$$\Delta\xi(t) = \xi((t+\Delta t) \wedge \theta_{U_m}) - \xi(t).$$

For $m' \in \mathcal{U}_m$ we can calculate the regression

$$Y^0(t,m')_{|U_m} = \lim_{\Delta t \downarrow 0} E\left[\left(\frac{\Delta\xi(t)}{\Delta t}\right) \mid \xi(t) = m'\right].$$

Construct the vector field $Y^0(t,\cdot)$, assigning to each $m \in M$ the corresponding vector $Y^0(t,m)|_{U_m}$ in $\mathcal{U}_m$. Thus by construction Y^0 is a measurable section of tangent bundle TM, i.e., a vector field.

Definition 4. *The random vector*

$$D^H\xi(t) = Y^0(t,\xi(t))$$

is called forward mean derivative of process $\xi(t)$ *on* M *at time instant* t *with respect to* $\mathcal{H}$.

In analogy with above construction we give the definition of backward mean derivative. For a relatively compact normal chart $\mathcal{U}_m$ of $\mathcal{H}$ containing a point $m \in M$ denote by $\overset{*}{\theta}_{\mathcal{U}_m}$ the random time of the last exit of $\xi(\cdot)$ from the boundary of $\mathcal{U}_m$ with $\xi(t) \in \mathcal{U}_m$. Introduce the notation

$$\Delta_*\xi(t) = \xi(t) - \xi((t-\Delta t) \vee \theta_{U_m}).$$

Then for $m' \in \mathcal{U}_m$ calculate the regression

$$Y^0_*(t,m')_{|U_m} = \lim_{\Delta t \downarrow 0} E\left[\left(\frac{\Delta_*\xi(t)}{\Delta t}\right) \mid \xi(t) = m'\right].$$

Construct the vector field $Y^0_*(t,\cdot)$, assigning to each $m \in M$ the corresponding vector $Y^0_*(t,m)|_{\mathcal{U}_m}$ in $\mathcal{U}_m$. Thus by construction Y^0_* is a measurable section of tangent bundle TM, i.e., a vector field.

Definition 5. *The random vector*

$$D^{\mathcal{H}}_*\xi(t) = Y^0_*(t,\xi(t))$$

is called beckward mean derivative of process $\xi(t)$ *on* M *at time instant* t *with respect to* $\mathcal{H}$.

Definition 6. *The limit*

$$D_2\xi(t) = \lim_{\Delta t \downarrow 0} E^\xi_t\left(\frac{\Delta\xi(t) \otimes \Delta\xi(t)}{\Delta t}\right) \qquad (10)$$

where $\otimes$ *is the tensor product in model space containing a chart, is called the quadratic mean derivative of* $\xi(t)$ *on* M *at time instant* t.

Note that the quadratic mean derivative is well-posed independently of any connection. It should be also pointed out that if we define an object by (10) where $\Delta\xi(t)$ is replaced by $\Delta_*\xi(t)$, for a solution of $(\hat{a}, A)$ we shall obtain $D_2\xi(t)$ as well.

The next statement follows from the results of [11—13].

Theorem 3. *Let the Itô vector field* $(a(t,m), A(t,m))$ *canonically correspond to an Itô equation* $(\hat{a}(t,m), A(t,m))$ *with respect to a connection* $\mathcal{H}$. *Then for a solution* $\xi(t)$ *of* $(\hat{a}(t,m), A(t,m))$ *we have:*

$$D^{\mathcal{H}}\xi(t) = a(t,\xi(t)),$$
$$D_2\xi(t) = \alpha(t,\xi(t)),$$

where $\alpha(t,m)$ *is the symmetric* $(2,0)$*-tensor field* $\alpha(t,m) = A(t,m)A^*(t,m)$, *and* $A^*(t,m)$ *is the conjugate operator to* $A(t,m)$.

Thus from Theorems 2 and 3 and from formula (2) it follows that for above $\xi(t)$ with generator L we have

$$D^{\mathcal{H}}\xi(t) = (\mathcal{H}L)(t,\xi(t)) \qquad (11)$$

and

$$D_2\xi(t) = 2(\mathcal{Q}L)(t,\xi(t)) \qquad (16)$$

where $\mathcal{H}$ is introduced in (3) and $\mathcal{Q}$ in (2).

If $\mathcal{H}$ is specified, we shall not indicate it in the notation of mean derivatives.

Remark 1. *Let* $f : M \to M_1$ *be a smooth mapping of manifolds. Note that, since the value of a mean derivative depends on the "now'' σ-algebra of the process, the tangent mapping* Tf *sends mean derivatives of a process* $\eta(t)$ *to mean derivatives of the process* $\xi(t) = f(\eta(t))$ *only in the following form:* $Tf(D\eta(t)) = D^\eta(\xi(t))$ *or* $Tf(D^\xi\eta(t)) = D\xi(t)$ *but generally speaking* $Tf(D\eta(t)) \neq D\xi(t)$. *Analogous fact in true for backward mean derivatives:* $Tf(D_*\eta(t)) = D^\eta_*(\xi(t))$ $Tf(D^\xi_*\eta(t)) = D_*\xi(t)$ *but generally speaking* $Tf(D_*\eta(t)) \neq D_*\xi(t)$.

Notice that if we apply the same connection both for transition from $(\hat{a}, A)$ to (a, A) and for determining the mean derivative, we obtain for a solution $\xi(t)$ that $D\xi(t) = a(t,\xi(t))$. Moreover, if we change the connection, the Itô vector filed (a, A) canonically corresponding to $(\hat{a}, A)$, and the forward mean derivative $D\xi(t)$ will be changed but the equality $D\xi(t) = a(t,\xi(t))$ for those new values will remain true.

Now let us turn to relations between backward mean derivatives and backward infinitesimal generators.

For the sake of simplicity, if $\xi(t)$ is a solution of $(\hat{a}, A)$ we rename the vector $Y^0_*(t,m)$ as $a_*(t,m)$, thus $D_*\xi(t) = a_*(t,\xi(t))$.

Introduce $D^\xi_* w(t)$ by formula $D^\xi_* w(t) = \lim_{\Delta t \to +0} E^\xi_t \frac{w(t) - w(t-\Delta t)}{\Delta t}$ where E^ξ_t is the conditional expectation with respect to the «present» σ-algebra of $\xi(t)$ (see above).

Definition 7 *The process* $w_*^{\xi}(t)=\int_t^T D_*^{\xi}w(s)ds+ +w(t)-w(T)$ *is called the backward Wiener process with respect to* $\xi(t)$.

Specify a connection $\mathcal{H}$ on M. Let (a,A) be an It ô vector field on M. Denote by ∇A the covariant derivative of the field $A(t,m)$ with respect to connection $\mathcal{H}$; ∇A is a field of bilinear operators $\nabla A(t,m)(\cdot,\cdot):T_mM\times\mathbb{R}^n\to T_mM$. Consider the field $\nabla A(t,m)(A\cdot,\cdot):\mathbb{R}^n\times\mathbb{R}^n\to T_mM$ and the related vector field

$$tr\nabla A(A)(t,m)=tr\nabla A(t,m)(A(t,m)(\cdot),\cdot).$$

Determine on M the following equation in the local coordinates of a chart $\mathcal{U}_\alpha$ as follows:

$$\begin{aligned} d\xi(t)=a(t,\xi(t))dt+tr\nabla A(A)(t,\xi(t))dt-\\ -A(t,\xi(t))\circ D_*^{\xi}w(t)dt- \qquad (12)\\ -\frac{1}{2}\mathbf{\Gamma}_{\xi(t)}(A,A)dt+A(t,\xi(t))dw(t).\end{aligned}$$

Direct verification shows that (12) transforms correctly (covariantly) under changes of coordinates. This means that equation (12) is well defined on entire M.

Theorem 4. *Let* $\xi(t)$, $\xi(0)=m_0$, *be a strong solution to* (12). *Then* $D_*\xi(t)=a(t,\xi(t))$ *for* $t\in(0,l]$.

Theorem 4 is proved as theorem 12.32 in [7].

Specify a certain time moment t. From the above formulae it follows that the process $\eta(t)$ such that $\xi(t)=\eta(t)$ and satisfying for $s<t$ the relation

$$\begin{aligned}\eta(t)-\eta(s)=\int_s^t a_*(\tau,\eta(\tau))d\tau+\\ +\int_s^t\frac{1}{2}\mathbf{\Gamma}_{\eta(\tau)}(A,A)d\tau+\int_s^t A(\tau,\eta(\tau))d_*w_*^{\xi}(\tau),\end{aligned} \qquad (13)$$

where $a_*(t,m)=a(t,m)-tr\nabla A(A\cdot,\cdot)+A(t,m)D_*^{\xi}w(t)$ and the last summand in the right-hand side is the backward stochastic integral, has the same backward mean derivative at t as $\xi(t)$.

Relation (13) is called Itô equation in backward differentials.

Thus for small enough $s<t$ such $\eta(s)$ approximates $\xi(s)$.

Introduce the notation $\hat{a}_*(t,m)=a_*(t,\xi(t))+ +\frac{1}{2}\mathbf{\Gamma}_{\xi(t)}(A,A)$. Taking into account the interrelations between the transformation rule for local connector and the second derivative of a change of coordinates $\varphi_{\beta\alpha}$ (see, e.g., [7]) we obtain that under the change of coordinates $\varphi_{\beta\alpha}$ between the charts $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ the triple $(m,(\hat{a}_*,A))$ transforms by the rule

$$(m^\alpha,(\hat{a}_*^\alpha,A^\alpha))\mapsto$$

$$\mapsto\varphi_{\beta\alpha}m^\alpha,\left(\varphi'_{\beta\alpha}\hat{a}_*^\alpha-\frac{1}{2}\mathrm{tr}\varphi''_{\beta\alpha}(A^\alpha,A^\alpha),\varphi'_{\beta\alpha}(A^\alpha)\right).$$

Thus $(\hat{a}_*,A)$ is a backward Itô equation according to Definition 2 (formula (5)).

Definition 8. *The Itô equation* $(\hat{a},A)$ *and the backward It ô equation* $(\hat{a}_*,A)$ *introduced above, are called coupled to each other.*

Denote by L_* the backward generator of $(\hat{a}_*,A)$ coupled with $(\hat{a},A)$ that describes the process $\xi(t)$. In local coordinates it obviously expressed in the form

$$L_*=-\hat{a}_*^i\frac{\partial}{\partial q^i}+(AA^*)^{ij}\frac{\partial^2}{\partial q^i\partial q^j}.$$

Theorem 5.

$$D_*^{\mathcal{H}}\xi(t)=-\mathcal{H}(L_*). \qquad (14)$$

Proof. By construction $D_*^{\mathcal{H}}\xi(t)=a_*(t,\xi(t))$ and

$$\begin{aligned}\hat{a}_*(t,\xi(t))=a_*+\frac{1}{2}\mathbf{\Gamma}_{\xi(t)}(A,A)=\\ =\hat{a}_*^k\frac{\partial}{\partial q^k}+\mathbf{\Gamma}_{ij}^k(AA_*)^{ij}\frac{\partial}{\partial q^k}.\end{aligned}$$

On the other hand, we obtain that $\mathcal{H}(L_*)=-\hat{a}_*^k\frac{\partial}{\partial q^k}+$

$$+\mathbf{\Gamma}_{ij}^k(AA^*)^{ij}\frac{\partial}{\partial q^k}=-a_*^k\frac{\partial}{\partial q^k}-\mathbf{\Gamma}_{ij}^k(AA_*)^{ij}\frac{\partial}{\partial q^k}+$$

$$+\mathbf{\Gamma}_{ij}^k(AA_*)^{ij}\frac{\partial}{\partial q^k}=-a_*\frac{\partial}{\partial q^k}.\ \blacksquare$$

Formula (14) is «symmetric» to (11).

REFERENCES

1. *Emery M.* Stochastic calculus on manifolds / M. Emery. — Berlin et al.: Springer, 1989.

2. *Meyer P.A.* A differential geometric formalism for the Ito calculus / P. A. Meyer // Lecture Notes in Mathematics, 1981. — Vol. 851. — P. 256—270.

3. *Schwartz L.* Semimartingales and their stochastic calculus on manifolds / L. Schwartz. — Montreal: Montreal University Press, 1984.

4. *Belopolskaya Ya.I.* Stochastic processes and differential geometry / Ya. I. Belopolskaya, Yu. L. Dalecky. Dordrecht: Kluwer Academic Publishers, 1989.

5. *Gliklikh Yu.E.* Ordinary and Stochastic Differential Geometry as a Tool for Mathematical Physics / Yu. E. Gliklikh. — Dordrecht: Kluwer, 1996.

6. *Gliklikh Yu.E.* Global Analysis in Mathematical Physics. Geometric and Stochastic Methods / Yu. E. Gliklikh. — N.Y.: Springer-Verlag, 1997.

7. *Gliklikh Yu.E.* Stochastic and global analysis in problems of mathematical physics / Yu. E. Gliklikh. — Moscow: KomKniga, 2005. (in Russian)

8. *Nelson E.* Derivation of the Schrödinger equation from Newtonian mechanics / E. Nelson // Phys. Reviews, 1966. — Vol. 150, № 4. — P. 1079—1085.

9. *Nelson E.* Dynamical theory of Brownian motion / E. Nelson. — Princeton: Princeton University Press, 1967.

10. *Nelson E.* Quantum fluctuations / E. Nelson. — Princeton: Princeton University Press, 1985

11. *Azarina S.V.* Differential equations and inclusions with forward mean derivatives in $\mathbb{R}^n$ / S. V. Azarina, Yu. E. Gliklikh // Proceedings of Voronezh State University, 2006. — № 2. — P. 138—146 (in Russian)

12. *Azarina S.V.* Differential inclusions with mean derivatives / S. V. Azarina, Yu. E. Gliklikh // Dynamic Systems and Applications. — 2007. — Vol. 16, № 1. — P. 49—71

13. *Azarina S.V.* On differential equations and inclusions with mean derivatives on a compact a manifold / S. V. Azarina, Yu. E. Gliklikh // Discussiones Mathematicae. DICO. — 2007. — Vol. 27. — № 2. — P. 385—397.

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