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ON THE INITIAL-BOUNDARY VALUE PROBLEM FOR EQUATIONS OF ANOMALOUS DIFFUSION IN POLYMERS

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We study the system of partial differential equations which describes the diffusion of a penetrant liquid in a polymer. We construct weak solutions to the initial-boundary value problem for this system in a bounded domain.

KEY WORDS: anomalous diffusion, polymer—penetrant systems, topological degree, weak solution, nonlinear PDE.

1. INTRODUCTION

It is well known that diffusion in continuums is described by the following conservation law:

$$\frac{\partial u}{\partial t} = -div\, J \qquad (1.1)$$

where $u = u(t,x)$ is the concentration and $J = J(t,x)$ is the concentration flux vector (they depend on time t and the spatial point x).

The classical Fick's law states that the flux is proportional to the concentration gradient:

$$J = -D(u)\nabla u \qquad (1.2)$$

where $D(u)$ is the diffusion coefficient (generally speaking, it is a positive-definite tensor). Formulas (1.1) and (1.2) yield the classical diffusion equation

$$\frac{\partial u}{\partial t} = div\,(D(u)\nabla u). \qquad (1.3)$$

If $D(u) \equiv DI$ (where I is the unit tensor and D is a positive number), then (1.3) becomes the heat equation

$$\frac{\partial u}{\partial t} = D\Delta u. \qquad (1.4)$$

Experiments show that the concentration behaviour in diffusion processes in polymers cannot be described by (1.3) or (1.4) (see e.g. [16]). Let us mention two examples of such phenomena. The first one is so-called "case II diffusion" where concentration fronts can move with constant speed (the Fick's law implies that a front should propagate with speed proportional to $\frac{1}{\sqrt{t}}$). The second one is called "sorption overshoot". It means that the mass of penetrant absorbed by the polymer increases sharply until some point and then decreases, little by little, to a steady-state value [3].

Thus, Fick's law (1.2) should be replaced by another relation in order to explain the observed phenomena. One of such relations (based on the relaxation (viscoelastic) mechanism) was proposed by Cohen et al. [3,4] for the diffusion of a penetrant liquid in a polymer:

$$J = -D(u)\nabla u - $$
$$-E(u)\nabla \int_{-\infty}^{t} \exp\left(\int_{t}^{s} \beta(u(\xi,x))\,d\xi\right) f\left(u(s,x), \frac{\partial u(s,x)}{\partial s}\right) ds. \qquad (1.5)$$

Here E, β, D are functions of a scalar argument, f is a scalar function of two scalar arguments, D and E are called the diffusion and stress-diffusion coefficients, respectively. The function β is the inverse of the relaxation time. A typical example of β is [3]:

$$\beta(u) = \frac{1}{2}(\beta_R + \beta_G) + \frac{1}{2}(\beta_R - \beta_G)\tanh\left(\frac{u - u_{RG}}{\delta}\right)$$

where $\beta_R, \beta_G, \delta, u_{RG}$ are positive constants, $\beta_R > \beta_G$.

The constitutive law (1.5) may be rewritten as a system of two differential equations using the new variable $\sigma(t,x) = \int_{-\infty}^{t} \exp\left(\int_{t}^{s} \beta(u(\xi,x))\,d\xi\right) f(u(s,x), \frac{\partial u(s,x)}{\partial s})\, ds$ which is called stress (but has no exact connection with the classical stress tensor):

$$J(t,x) = -D(u)\nabla u - E(u)\nabla\sigma, \qquad (1.6)$$

$$\frac{\partial \sigma}{\partial t} + \beta(u)\sigma = f\left(u, \frac{\partial u}{\partial t}\right). \qquad (1.7)$$

Then (1.6) and (1.1) yield the diffusion equation:

$$\frac{\partial u}{\partial t} = div[D\nabla u + E\nabla\sigma]. \qquad (1.8)$$

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Generally speaking [2], D and E depend on t, x, u and σ.

A typical (but simple) form for the function f is

$$f(u, u') = \mu u + \nu u' \qquad (1.9)$$

where μ and ν are constants. This relation was used, for instance, in [4, 5, 14] (and in [3] with $\nu = 0$). In this paper we assume, however, that μ is not a constant.

Initial-boundary value problems for some particular cases of the general system (1.1), (1.6), (1.7) were studied in [1, 2, 7]. H. Amann [2] considers a wide class of these particular cases and shows existence of maximal (not global in time) solutions. A global existence result is given in [7] but under additional conditions on initial and boundary data. Another result on global in time solvability is presented in [1] for $f = \mu u$ and $D = E = const$. It is formulated for $0 < x < 1$, but the technique used there seems to be applicable for $x \in \Omega$ where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Global existence of dissipative (ultra weak) solutions was shown in [17] for $\Omega = \mathbb{R}^n$ (again, the ideas used there seem to be suitable for $\Omega \subset \mathbb{R}^n$ also).

In [11], a slightly different model of this kind was studied.

2. NOTATIONS

We use the standard notations $L_p(\Omega)$, $W_p^m(\Omega)$, $H^m(\Omega) = W_2^m(\Omega)$ $(m \in Z, 1 \le p \le \infty)$, $H_0^m(\Omega) = \overset{\circ}{W}{}_2^m(\Omega)$ $(m \in \mathbb{N})$ for Lebesgue and Sobolev spaces of functions defined on a bounded open set (domain) $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$.

The scalar product and the Euclidian norm in $L_2(\Omega)^k = L_2(\Omega, \mathbb{R}^k)$, are denoted by (u, v) and $\|u\|$, respectively (k is equal to 1 or n). In $H_0^1(\Omega)$, we use the following scalar product and norm: $(u, v)_1 = (\nabla u, \nabla v), \|u\|_1 = \|\nabla u\|$. We recall Friedrichs' inequality

$$\|u\| \le K_\Omega \|u\|_1. \qquad (2.1)$$

Similarly, in $H_0^2(\Omega)$, we use the scalar product and norm: $(u, v)_2 = (\Delta u, \Delta v), \|u\|_2 = \|\Delta u\|$.

As usual, we identify the space $H^{-m}(\Omega)$, $m = 1, 2$, with the space of linear continuous functionals on $H_0^m(\Omega)$ (the dual space).

Sometimes we shall write simply L_p, H^m for $L_p(\Omega)^k, H^m(\Omega)^k$ etc., $k = 1, n$.

The Laplace operator $\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism. Therefore

$$\Delta^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega) \qquad (2.2)$$

is also an isomorphism. Set $X = X(\Omega) = \Delta^{-1}(H_0^1(\Omega))$. The scalar product and norm in X are $(u, v)_X = (\Delta u, \Delta v)_1$, $\|u\|_X = \|\Delta u\|_1$.

The symbols $C(\mathcal{J}; E)$, $C_w(\mathcal{J}; E)$, $L_2(\mathcal{J}; E)$ etc. denote the spaces of continuous, weakly continuous, quadratically integrable etc. functions on an interval $\mathcal{J} \subset \mathbb{R}$ with values in a Banach space E. We recall that a function $u : \mathcal{J} \to E$ is *weakly continuous* if for any linear continuous functional g on E the function $g(u(\cdot)) : \mathcal{J} \to \mathbb{R}$ is continuous.

If E is a function space ($L_2(\Omega), H^m(\Omega)$ etc.), then we identify the elements of $C(\mathcal{J}; E)$, $L_2(\mathcal{J}; E)$ etc. with scalar functions defined on $\mathcal{J} \times \Omega$ according to the formula

$$u(t)(x) = u(t, x), t \in \mathcal{J}, x \in \Omega.$$

We shall also use the function spaces (T is a positive number):

$$W = W(\Omega, T) =$$
$$= \{\tau \in L_2(0, T; H_0^1(\Omega)), \tau' \in L_2(0, T; H^{-1}(\Omega))\}$$
$$\|\tau\|_W = \|\tau\|_{L_2(0,T;H_0^1(\Omega))} + \|\tau'\|_{L_2(0,T;H^{-1}(\Omega))};$$
$$W_1 = W_1(\Omega, T) =$$
$$= \{\tau \in L_2(0, T; X(\Omega)), \tau' \in L_2(0, T; H^{-1}(\Omega))\}$$
$$\|\tau\|_{W_1} = \|\tau\|_{L_2(0,T;X(\Omega))} + \|\tau'\|_{L_2(0,T;H^{-1}(\Omega))};$$
$$W_2 = W_2(\Omega, T) =$$
$$= \{\tau \in L_2(0, T; H_0^2(\Omega)), \tau' \in L_2(0, T; H^{-2}(\Omega))\}$$
$$\|\tau\|_{W_2} = \|\tau\|_{L_2(0,T;H_0^2(\Omega))} + \|\tau'\|_{L_2(0,T;H^{-2}(\Omega))}.$$

Lemma III.1.2 from [15] implies continuous embeddings $W, W_2 \subset C([0, T]; L_2(\Omega))$, $W_1 \subset \subset C([0, T]; H_0^1(\Omega))$ (see also [6]).

Denote by $\mathbb{R}^{n\times n}$ the space of matrices of the order $n \times n$ with the norm

$$\|A\|_{\mathbb{R}^{n\times n}} = \max_{\|\xi\|_{\mathbb{R}^n}=1} \|A\xi\|_{\mathbb{R}^n}.$$

Let $\mathbb{R}_+^{n\times n} \subset \mathbb{R}^{n\times n}$ be the set of such matrices A that

$$(A\xi, \xi)_{R^n} \ge d(A)(\xi, \xi)_{R^n}$$

for some $d(A) > 0$ and all $\xi \in \mathbb{R}^n$.

3. WEAK FORMULATION OF THE PROBLEM

We consider a polymer filling a bounded domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$. The most important particular cases are $n = 2$ (diffusion in polymer films) and $n = 3$. We study the diffusion of a penetrant in this polymer which is described by the following initial-boundary value problem:

$$\frac{\partial u}{\partial t} = div[D_0(t,x,u,\sigma)\nabla u + E_0(t,x,u,\sigma)\nabla\sigma], \quad (t,x)\in[0,T]\times\Omega, \tag{3.1}$$

$$\frac{\partial\sigma}{\partial t} + \beta_0(t,x,u,\sigma)\sigma = \mu_0(t,x,u,\sigma)u + \nu_0\frac{\partial u}{\partial t}, \quad (t,x)\in[0,T]\times\Omega, \tag{3.2}$$

$$u(t,x) = \varphi(t,x), (t,x)\in[0,T]\times\partial\Omega, \tag{3.3}$$

$$u(0,x) = u_0(x), \sigma(0,x) = \sigma_0(x), x\in\Omega. \tag{3.4}$$

Here $u = u(t,x):[0,T]\times\overline{\Omega}\to\mathbb{R}$ is the unknown concentration of the penetrant (at the spatial point x at the moment of time t), $\sigma = \sigma(t,x):[0,T]\times\overline{\Omega}\to\mathbb{R}$ is the unknown stress, $u_0 = u_0(x)$, $\sigma_0 = \sigma_0(x):\Omega\to\mathbb{R}$ are given initial data, $\varphi:[0,T]\times\partial\Omega\to\mathbb{R}$ is a given boundary condition, ν_0 is a given positive constant, D_0, $E_0:\mathbb{R}^{n+3} = \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^{n\times n}_+$, μ_0, $\beta_0:\mathbb{R}^{n+3}\to\mathbb{R}$ are given functions.

In order to come to a definition of a weak solution to this problem, let us carry out some changes of variables and other heuristic operations. For this purpose, assume for a while that all the functions (given and unknown) involved in the equations and the domain Ω are sufficiently regular. Suppose also that

$$|u(t,x)|\le 1 \tag{3.5}$$

(remember that u is the concentration, so it cannot exceed 100 %). W.l.o.g. below we assume that the function φ is defined on $[0,T]\times\overline{\Omega}$, satisfies the same estimate as (3.5) and $\varphi|_{t=0} = u_0$ (cf. [9, Theorem 1.1]).

Denote $\varsigma = \sigma - \nu_0 u$, $\varsigma_0 = \sigma_0 - \nu_0 u_0$, $\gamma(\cdot,\cdot,u,\varsigma) = \mu_0(\cdot,\cdot,u,\varsigma+\nu_0 u) - \nu_0\beta_0(\cdot,\cdot,u,\varsigma+\nu_0 u)$, $D_1(t,x,u,\varsigma) = D_0(t,x,u,\varsigma+\nu_0 u) + \nu_0 E_0(t,x,u,\varsigma+\nu_0 u)\in\mathbb{R}^{n\times n}_+$, $E_1(\cdot,\cdot,u,\varsigma) = E_0(\cdot,\cdot,u,\varsigma+\nu_0 u)$, $\beta_1(\cdot,\cdot,u,\varsigma) = -\beta_0(\cdot,\cdot,u,\varsigma+\nu_0 u)$. Then we can rewrite (3.1), (3.2) and (3.4) in the following form:

$$\frac{\partial u}{\partial t} = div[D_1(t,x,u,\varsigma)\nabla u + E_1(t,x,u,\varsigma)\nabla\varsigma], \tag{3.6}$$

$$\frac{\partial\varsigma}{\partial t} = \beta_1(t,x,u,\varsigma)\varsigma + \gamma(t,x,u,\varsigma)u, \tag{3.7}$$

$$u|_{t=0} = u_0, \varsigma|_{t=0} = \varsigma_0. \tag{3.8}$$

Equations (3.7) and (3.8) yield

$$\varsigma(t,x) = \varsigma_0(x)\exp\left(\int_0^t \beta_1(\xi,x,u(\xi,x),\varsigma(\xi,x))\,d\xi\right) + \int_0^t \exp\left(\int_s^t \beta_1(\xi,x,u(\xi,x),\varsigma(\xi,x))\,d\xi\right)\times \gamma(s,x,u(s,x),\varsigma(s,x))u(s,x)\,ds. \tag{3.9}$$

Thus, if $|\varsigma_0(x)|$ is uniformly bounded, ς is also uniformly bounded by a constant dependent on T:

$$|\varsigma(t,x)|\le K_T. \tag{3.10}$$

Now, for each fixed $x\in\overline{\Omega}$, consider the Cauchy problem

$$\frac{\partial\psi}{\partial t} = \beta_1(t,x,\varphi,\psi)\psi + \gamma(t,x,\varphi,\psi)\varphi, \tag{3.11}$$

$$\psi|_{t=0} = \varsigma_0. \tag{3.12}$$

The solution $\psi(t,x)$ of this problem is a priori bounded by K_T. Therefore it exists and is unique on the whole segment $[0,T]$. Observe that $\varsigma|_{\partial\Omega} = \psi|_{\partial\Omega}$.

Apply the Laplace operator to both sides of (3.7):

$$\Delta\varsigma' = div[\nabla(\beta_1(t,x,u,\varsigma)\varsigma) + \nabla(\gamma(t,x,u,\varsigma)u)]. \tag{3.13}$$

Hence,

$$\Delta\varsigma' = div[\beta_1(t,x,u,\varsigma)\nabla\varsigma + \frac{\partial\beta_1}{\partial x}(t,x,u,\varsigma)\varsigma + \frac{\partial\beta_1}{\partial u}(t,x,u,\varsigma)\varsigma\nabla u + \frac{\partial\beta_1}{\partial\varsigma}(t,x,u,\varsigma)\varsigma\nabla\varsigma + \gamma(t,x,u,\varsigma)\nabla u + \frac{\partial\gamma}{\partial x}(t,x,u,\varsigma)u + \frac{\partial\gamma}{\partial u}(t,x,u,\varsigma)u\nabla u + \frac{\partial\gamma}{\partial\varsigma}(t,x,u,\varsigma)u\nabla\varsigma]. \tag{3.14}$$

Let $\chi:\mathbb{R}\to\mathbb{R}$ be a bounded continuous function such that $\chi(s) = s$ for $|s| < 1 + K_T$. Then

$$\Delta\varsigma' = div[\frac{\partial\beta_1}{\partial u}(t,x,u,\varsigma)\chi(\varsigma)\nabla u + \gamma(t,x,u,\varsigma)\nabla u + \frac{\partial\gamma}{\partial u}(t,x,u,\varsigma)\chi(u)\nabla u + \beta_1(t,x,u,\varsigma)\nabla\varsigma + \frac{\partial\beta_1}{\partial\varsigma}(t,x,u,\varsigma)\chi(\varsigma)\nabla\varsigma + \frac{\partial\gamma}{\partial\varsigma}(t,x,u,\varsigma)\chi(u)\nabla\varsigma + \frac{\partial\beta_1}{\partial x}(t,x,u,\varsigma)\varsigma + \frac{\partial\gamma}{\partial x}(t,x,u,\varsigma)u]. \tag{3.15}$$

Denote $v = u - \varphi$, $\tau = \varsigma - \psi$, $\beta(t,x,v,\tau) = \frac{\partial\beta_1}{\partial u}(t,x,v+\varphi,\tau+\psi)\chi(\tau+\psi) + \gamma(t,x,v+\varphi,\tau+\psi) + \frac{\partial\gamma}{\partial u}(t,x,v+\varphi,\tau+\psi)\chi(v+\varphi)$, $\mu(t,x,v,\tau) = \beta_1(t,x,v+\varphi,\tau+\psi) + \frac{\partial\beta_1}{\partial\varsigma}(t,x,v+\varphi,\tau+\psi)\chi(\tau+\psi) + \frac{\partial\gamma}{\partial\varsigma}(t,x,v+\varphi,\tau+\psi)\chi(v+\varphi)$, $g(t,x,v,\tau) = -\nabla\psi' + \frac{\partial\beta_1}{\partial u}(t,x,v+\varphi,\tau+\psi)\chi(\tau+\psi)\nabla\varphi + \gamma(t,x,v+\varphi,\tau+\psi)\nabla\varphi + \frac{\partial\gamma}{\partial u}(t,x,v+\varphi,\tau+\psi)\chi(v+\varphi)\nabla\varphi + \beta_1(t,x,v+\varphi,\tau+\psi)\nabla\psi + \frac{\partial\beta_1}{\partial\varsigma}(t,x,v+\varphi,\tau+\psi)\chi(\tau+\psi)\nabla\psi + \frac{\partial\gamma}{\partial\varsigma}(t,x,v+\varphi,\tau+\psi)\chi(v+\varphi)\nabla\psi + \frac{\partial\beta_1}{\partial x}(t,x,v+\varphi,\tau+\psi)(\tau+\psi) + \frac{\partial\gamma}{\partial x}(t,x,v+\varphi,\tau+\psi)(v+\varphi)$.

Then we can rewrite (3.15) in the following form:

$$\Delta\tau' = = div[\beta(t,x,v,\tau)\nabla v + \mu(t,x,v,\tau)\nabla\tau + g(t,x,v,\tau)]. \quad (3.16)$$

Similarly, using the notations $D(t,x,v,\tau) = D_1(t,x,v+\varphi,\tau+\psi)$, $E(t,x,v,\tau) = E_1(t,x,v+\varphi,\tau+\psi)$, $f(t,x,v,\tau) = -\nabla\Delta^{-1}\varphi' + D_1(t,x,v+\varphi,\tau+\psi)\nabla\varphi + E_1(t,x,v+\varphi,\tau+\psi)\nabla\psi$, we rewrite (3.6) as

$$v' = = div[D(t,x,v,\tau)\nabla v + E(t,x,v,\tau)\nabla\tau + f(t,x,v,\tau)]. \quad (3.17)$$

Observe that the initial and boundary conditions for v and τ are

$$v|_{t=0} = 0, \tau|_{t=0} = 0, \quad (3.18)$$

$$v|_{\partial\Omega} = 0, \tau|_{\partial\Omega} = 0. \quad (3.19)$$

Definition 3.1. A pair of functions (v,τ) from the class

$$v \in W(\Omega,T), \tau \in H^1(0,T;H_0^1(\Omega)) \quad (3.20)$$

is a *weak* solution to problem (3.16)—(3.19) if it satisfies (3.18) and equalities (3.16), (3.17) hold in the space $H^{-1}(\Omega)$ a.e. on $(0,T)$.

Note that (3.18) makes sense due to the embeddings

$$W \subset C([0,T];L_2(\Omega)), H^1(0,T;H_0^1(\Omega)) \subset \subset C([0,T];H_0^1(\Omega)).$$

Condition (3.19) is "hidden" in the space $H_0^1(\Omega)$.

As we have seen, for any regular solution (u,σ) of the problem (3.1)—(3.4) satisfying (3.5), the corresponding pair (v,τ) is a weak solution to problem (3.16)—(3.19). Conversely, let (v,τ) be a weak solution to problem (3.16)—(3.19). Assume that the functions v,τ,ψ,φ are sufficiently regular, (3.11) holds, and the corresponding pair $(u = v+\varphi, \varsigma = \tau+\psi)$ satisfies (3.5) and (3.10). Then we have (3.6) and (3.13), and the latter gives (3.7) due to (3.11). Moreover, we have (3.8) with $u_0 = \varphi|_{t=0}$, $\varsigma_0 = \psi|_{t=0}$. Thus, $(u, \sigma = \varsigma + \nu_0 u)$ is a solution to (3.1)—(3.4).

When dealing with weak solutions, it is not necessary to assume that the functions in (3.16)—(3.19) and Ω are so regular as it was above. Let us describe the conditions which we impose. It is not hard to see that these conditions are fulfilled provided the functions in (3.1)—(3.4) and Ω are sufficiently regular (as it was above).

For the sake of generality we assume that the functions β and μ are matrix-valued and replace (3.18) with more general initial condition

$$v|_{t=0} = v_0 \in L_2, \tau|_{t=0} = \tau_0 \in H_0^1 \quad (3.21)$$

(with the corresponding change in Definition 3.1).

Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be any bounded open set such that

$$X(\Omega) \subset W_{p_0}^1(\Omega) \quad (3.22)$$

for some $p_0 > 2$.

Assume that

i) D, E, μ, $\beta : \mathbb{R}^{n+3} \to \mathbb{R}^{n\times n}$; $f,g : \mathbb{R}^{n+3} \to \mathbb{R}^n$.

ii) Each of these six functions (e.g. $D(t,x,v,\tau)$) is measurable in (t,x) for fixed (v,τ).

iii) Each of these functions is continuous in (v,τ) for fixed (t,x).

iv) These functions satisfy the estimates

$$|D(t,x,v,\tau)|_{R^{n\times n}} \le K_D, \quad (3.23)$$

$$|E(t,x,v,\tau)|_{R^{n\times n}} \le K_E, \quad (3.24)$$

$$|\beta(t,x,v,\tau)|_{R^{n\times n}} \le K_\beta, \quad (3.25)$$

$$|\mu(t,x,v,\tau)|_{R^{n\times n}} \le K_\mu, \quad (3.26)$$

$$|f(t,x,v,\tau)|_{R^n} \le K_f(|v|+|\tau|) + \tilde{f}(t,x), \quad (3.27)$$

$$|g(t,x,v,\tau)|_{R^n} \le K_g(|v|+|\tau|) + \tilde{g}(t,x) \quad (3.28)$$

with some constants $K_D,\ldots,K_g$ and functions $\tilde{f},\tilde{g} \in L_2((0,T)\times\Omega)$.

v)

$$(D(t,x,v,\tau)\xi,\xi)_{R^n} \ge d(\xi,\xi)_{R^n}, \quad (3.29)$$

where $d > 0$ is independent of $(t,x,v,\tau) \in \mathbb{R}^{n+3}$ and $\xi \in \mathbb{R}^n$.

Theorem 3.1. *Under these conditions, for every $v_0 \in L_2(\Omega)$ and $\tau_0 \in H_0^1(\Omega)$, there exists a weak solution to problem (3.16), (3.17), (3.19), (3.21) in class (3.20).*

4. AUXILIARY PROBLEM

Consider the following auxiliary problem:

$$\frac{\partial v}{\partial t} + \varepsilon\Delta^2 v = \lambda div[D(t,x,v,\tau)\nabla v + +E(t,x,v,\tau)\nabla\tau + f(t,x,v,\tau)], \quad (4.1)$$

$$\frac{\partial \tau}{\partial t} + \varepsilon\Delta^2 \tau = \lambda\Delta^{-1} div[\beta(t,x,v,\tau)\nabla v + +\mu(t,x,v,\tau)\nabla\tau + g(t,x,v,\tau)], \quad (4.2)$$

$$v|_{t=0} = v_0, \quad (4.3)$$

$$\tau|_{t=0} = \tau_0. \quad (4.4)$$

Here $\varepsilon > 0$, $\lambda \in [0,1]$ are parameters.

Definition 4.1. Given $v_0 \in L_2(\Omega)$, $\tau_0 \in H_0^1(\Omega)$, a pair of functions (v,τ) from the class

$$v \in W_2(\Omega, T), \tau \in W_1(\Omega, T) \qquad (4.5)$$

is a *weak* solution of problem (4.1)—(4.4) if equality (4.1) holds in the space $H^{-2}(\Omega)$ a.e. on $(0,T)$, (4.2) holds in the space $H^{-1}(\Omega)$ a.e. on $(0,T)$, (4.3) holds in $L_2(\Omega)$, and (4.4) holds in $H_0^1(\Omega)$.

The last two conditions make sense due to the embeddings

$$W_1 \subset C([0,T]; H_0^1(\Omega)), W_2 \subset C([0,T]; L_2(\Omega)).$$

The proof of Theorem 3.1 is based on three lemmas.

Lemma 4.1. *Let (v,τ) be a weak solution to problem (4.1)—(4.4). Then the following a priori estimate holds:*

$$\varepsilon \|v\|^2_{L_2(0,T;H_0^2(\Omega))} + \varepsilon \|\tau\|^2_{L_2(0,T;X)} + \|v\|^2_{L_\infty(0,T;L_2(\Omega))} + $$
$$+\lambda \|v\|^2_{L_2(0,T;H_0^1(\Omega))} + \|\tau\|^2_{L_\infty(0,T;H_0^1(\Omega))} \le C \qquad (4.6)$$

where C is independent of λ and ε.

Lemma 4.2. *Let (v,τ) be a weak solution to problem (4.1)—(4.4). Then there is the following bound of the time derivatives:*

$$\|v'\|_{L_2(0,T;H^{-2}(\Omega))} + \|\tau'\|_{L_2(0,T;H^{-1}(\Omega))} \le C(1+\sqrt{\varepsilon}) \quad (4.7)$$

where C is independent of λ and ε.

Lemma 4.3. *Given $v_0 \in L_2(\Omega)$, $\tau_0 \in H_0^1(\Omega)$, there exists a weak solution to problem (4.1)—(4.4) in class (4.5).*

The proof of this lemma is based on degree theory arguments, and then Theorem 3.1 follows via passage to the limit as $\varepsilon \to 0$ (cf. [18, 19]).

REFERENCES

1. *Amann H.* Global existence for a class of highly degenerate parabolic systems. Japan J. Indust. Appl. Math., 1991, V. 8, P. 143—151.

2. *Amann H.* Highly degenerate quasilinear parabolic systems, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 1991, V. 18, P. 135—166.

3. *Cohen D. S., White A. B., Jr., Witelski T. P.* Shock formation in a multidimensional viscoelastic diffusive system, SIAM J. Appl. Math., 1995, V. 55, №. 2, P. 348—368.

4. *Edwards D. A., Cohen D. S.* A mathematical model for a dissolving polymer, AIChE J., 1995, V. 18, P. 2345—2355.

5. *Edwards D. A.* A mathematical model for trapping skinning in polymers, Studies in Applied Mathematics, 1997, V. 99, P. 49—80.

6. *Gajewski H., Groeger K., Zacharias K.* Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen, Akademie-Verlag, Berlin, 1974.

7. *Hu B., Zhang J.* Global existence for a class of non-Fickian polymer-penetrant systems. J. Partial Diff. Eqs., 1996, V. 9, P. 193—208.

8. *Krasnoselskii M.* Topological methods in the theory of nonlinear integral equations, Gostehizdat, 1956, (Russian); Engl. transl., Macmillan, 1964.

9. *Krasnosel'skii M. A., Zabreiko P. P.* Geometrical methods of nonlinear analysis. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 263. Springer-Verlag, Berlin, 1984.

10. *Lloyd N. G.* Degree theory. Cambridge University Press, 1978.

11. *Riviere B., Shaw S.* Discontinuous Galerkin finite element approximation of nonlinear non-Fickian diffusion in viscoelastic polymers, SIAM Journal on Numerical Analysis, 2006, V. 44. № 6, P. 2650—2670.

12. *Simon J.* Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura Appl, 1987, V. 146, P. 65—96.

13. *Skrypnik I. V.* Methods for analysis of nonlinear elliptic boundary value problems, Translations of Mathematical Monographs 139 (Amer. Math. Soc., 1994).

14. *Swaminathan S., Edwards D.A.* Travelling waves for anomalous diffusion in polymers, Applied Mathematics Letters, 2004, V. 17, P. 7—12.

15. *Temam R.* Navier-Stokes equations. Theory and numerical analysis. Studies in Mathematics and its Applications, Vol. 2. North-Holland Publishing Co., Amsterdam—New York—Oxford, 1977.

16. *Thomas N., Windle A. H.* Transport of methanol in poly-(methyl-methocry-late). Polymer 19 (1978) P. 255—265.

17. *Vorotnikov D. A.* Dissipative solutions for equations of viscoelastic diffusion in polymers, J. Math. Anal. Appl., 2008, Volume 339, P. 876—888.

18. *Zvyagin V. G., Dmitrienko V. T.* Approximating-topological approach to investigation of problems of hydrodynamics. Navier—Stokes system, Editorial URSS, Moscow, 2004.

19. *Zvyagin V. G., Vorotnikov D. A.* Topological approximation methods for evolutionary problems of nonlinear hydrodynamics, Walter de Gruyter, Berlin—New York, 2008.