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REGULARITY OF SOLUTION OF THE SECOND INITIAL BOUNDARY VALUE PROBLEM FOR PARABOLIC EQUATIONS IN DOMAINS WITH CONICAL POINTS

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The purpose of this paper is to establish the well-posedness and the regularity of solutions of the second initial boundary value problems for general higher order parabolic equations in infinite cylinders with the bases containing conical points.

KEY WORDS: parabolic equation, initial boundary value problem, nonsmooth domains, generalized solutions, regularity.

1. INTRODUCTION

We are concerned with initial boundary value problems for parabolic equations in nonsmooth domains. These problems with Dirichlet boundary condition in domains containing conical points have been investigated in [6, 7]. The problems with Neumann boundary condition in domains with edges have been dealt with for the classical heat equation in [10] and for general second order parabolic equations in [2]. In the present paper, we consider such problems with Neumann boundary condition (the second initial boundary problems) for higher order linear parabolic equations in domains containing conical points.

The main goal of this paper is to obtain the regularity of the solutions of the problems. There are some approaches to this issue. For parabolic equations of second order in a smooth domains it were established in both Hölder and Sobolev spaces in iteL by the method in which a regularizer was constructed and exact estimates of solutions in terms of the data of the problems were dealt with. Such ideas were also used in [2] with some modifications for the case of domains with edges. For theequation dealt with in [10], whose coefficients are independent of the time variable, one used Fourier transform to reduce the problem to an elliptic one with a parameter. In the present paper, for a general higher order linear parabolic equation with coefficients depending on both spatial and time variables in domains containing conical points we modify the approach suggested in [3, 6, 7]. First, we study the unique solvability and the regularity with respect to the time variable for generalized solutions in the Sobolev space $H^{m,1}(Q)$ by Galerkin's approximate method. By modifying the arguments used in [6, 7], we can weaken the restrictions on the data at the initial time $t=0$ imposed therein. After that, we take the term containing the derivative in time of the unknown function to the right-hand side of the equation such that the problem can be considered as an elliptic one. With the help of some auxiliary results we can apply the results for elliptic boundary value problems and our previous ones to deal with the regularity with respect to both of time and spatial variables of the solutions.

Our paper is organized as follow. In Sec. 2, we introduce some notations and the formulation of the problem. In Sec. 3 we establish the unique existence and the regularity with respect to time variable of the generalized solutions of the problem with the main result stated in Theorem 1. Theorem 2 is the main result of Sec. 4 in which the global regularity is dealt with.

2. NOTATION AND FORMULATION OF THE PROBLEM

Let G be a bounded domain in $\mathbb{R}^n (n \geq 2)$ with the boundary ∂G. We suppose that $\Gamma = \partial G \setminus \{0\}$ is a smooth manifold and G in a neighborhood of the origin 0 coincides with the cone $K = \{x : x/|x| \in \Omega\}$ where Ω is a smooth domain on the unit sphere S^{n-1} in $\mathbb{R}^n$. Set $Q_t = G \times (0,t)$ for each $t \in (0,+\infty), Q = Q_\infty = G \times (0,+\infty)$, and $S = \Gamma \times [0,+\infty)$. For each multi-index $\alpha = (\alpha_1, ts, \alpha_n) \in \mathbb{N}^n$, set $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $D^\alpha = D_x^\alpha = D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n}, D_{x_j} = -i\partial/\partial x_j$.

Let l be a nonnegative integer. We denote by $H^l(G)$ the usual Sobolev space of functions defined in G with the norm

$$\|u\|_{H^l(G)} = (\int_G \sum_{|\alpha|\leq m} \left|D^\alpha u\right|^2 dx)^{\frac{1}{2}},$$

by $H^{l-\frac{1}{2}}(\Gamma)$ the space of traces of functions from $H^l(G)$ on Γ with the norm

$$\|u\|_{H^{l-\frac{1}{2}}(\Gamma)} = \inf\{\|v\|_{H^l(G)} : v \in H^l(G), v|_\Gamma = u\}.$$

Let

$$L = L(x,t,D) = \sum_{|\alpha|,|\beta|=0}^{m} D^\alpha (a_{\alpha\beta}(x,t) D^\beta)$$

be a differential operator of order $2m$ defined in Q with coefficients infinitely differentiable up to the boundary, and let

$$B_j = B_j(x,t,D) = \sum_{|\alpha|\le m_j} b_\alpha(x,t) D^\alpha,\ j = 1,\dots,m,$$

be a system of boundary operators on S with coefficients infinitely differentiable in a neighborhood of $\partial G \times [0,+\infty), m \le \operatorname{ord} B_j = m_j \le 2m-1$ for $j = 1,\dots,m$. We assume that coefficients of L and B_j together with all derivatives are bounded in $\overline{Q}, \partial G \times [0,+\infty)$, respectively.

We assume that, for each $t \in [0,+\infty)$, $\{B_j(x,t,D)\}_{j=1}^m$ is a normal system on Γ (for the definition, see [5, Def. 3.14]). Then the following Green's formula

$$\int_G Lu\bar{v}dx = B(t,u,v) + \sum_{j=1}^{m} \int_\Gamma B_j u \overline{B'_j v} ds \quad (2.1)$$

is valid for all $u, v \in C_0^\infty(\overline{G} \setminus \{0\})$ and a.e. $t \in [0,+\infty)$, where $B'_j, j = 1,\dots,m,$ are boundary operators of order $2m-1-m_j$ on S, and

$$B(t,u,v) = \sum_{|\alpha|,|\beta|=0}^{m} \int_G a_{\alpha\beta}(.,t) D^\beta u \overline{D^\alpha v} dx, t \in [0,+\infty).$$

We also suppose that the form $B(t,.,.)$ is $H^m(G)$-elliptic uniformly with respect to $t \in [0,+\infty)$, i.e. the inequality

$$B(t,u,u) \ge \mu \|u\|^2_{H^m(G)} \quad (2.2)$$

is valid for all $u \in H^m(G)$ and all $t \in [0,+\infty)$, where μ is a positive constant independent of u and t.

We proceed to introduce some functional spaces. We introduce the spaces $V^l_{2,\gamma}(G)$, $H^l_\gamma(G)$ $(\gamma \in \mathbb{R})$ of functions in G equipped with the norms

$$\|u\|_{V^l_{2,\gamma}(G)} = (\sum_{|\alpha|\le l} \int_G r^{2(\gamma+|\alpha|-l)} |D^\alpha u|^2 dx)^{\frac{1}{2}},$$

$$\|u\|_{H^l_\gamma(G)} = (\sum_{|\alpha|\le l} \int_G r^{2\gamma} |D^\alpha u|^2 dx)^{\frac{1}{2}}.$$

For $l = 0$ we set $L_{2,\gamma}(G) = V^0_{2,\gamma}(G) = H^0_\gamma(G)$. If $l \ge 1$, then $V_\gamma^{l-\frac{1}{2}}(\Gamma), H_\gamma^{l-\frac{1}{2}}(\Gamma)$ denote the spaces consisting of traces of functions from respective spaces $V^l_{2,\gamma}(G), H^l_\gamma(G)$ on the boundary Γ with the respective norms

$$\|u\|_{V_\gamma^{l-\frac{1}{2}}(\Gamma)} = \inf\{\|v\|_{V^l_{2,\gamma}(G)} : v \in V^l_{2,\gamma}(G), v|_\Gamma = u\},$$

$$\|u\|_{H_\gamma^{l-\frac{1}{2}}(\Gamma)} = \inf\{\|v\|_{H^l_\gamma(G)} : v \in H^l_\gamma(G), v|_\Gamma = u\}.$$

It is obvious that $V^l_{2,\gamma}(G)$ is continuously imbedded in $H^l_\gamma(G)$. Moreover, for functions $u \in V^l_{2,\gamma}(G)$ the norms of u in $V^l_{2,\gamma}(G)$ and $H^l_\gamma(G)$ are equivalent. It is the same for $V_{gamma}^{l-\frac{1}{2}}(\Gamma)$ and $H_\gamma^{l-\frac{1}{2}}(\Gamma)$ (see [5, Th. 7.1.1, 7.1.2]). We have continuous imbeddings (see [5, Ch. 6, 7])

$$V^l_{2,\gamma}(G) \subset V^{l-k}_{2,\gamma-k}(G), H^l_\gamma(G) \subset H^{l-k}_{\gamma-k}(G) \quad \text{for } 0 \le k \le l \quad (2.3)$$

and

$$V_{2,\gamma}^{l-\frac{1}{2}}(\Gamma) \subset V_{2,\gamma-k}^{l-k-\frac{1}{2}}(\Gamma), H_\gamma^{l-\frac{1}{2}}(\Gamma) \subset H_{\gamma-k}^{l-k-\frac{1}{2}}(\Gamma) \quad \text{for } 0 \le k < l. \quad (2.4)$$

Let X, Y be Banach spaces. We denote by $L_2(0,T;X)$ $(0 < T \le +\infty)$ the space consisting of all measurable functions $u : (0,T) \to X$ with the norm

$$\|u\|_{L_2(0,T;X)} = (\int_0^T \|u(t)\|_X^2 dt)^{\frac{1}{2}},$$

and by $H^1(0,T;X,Y)$ the space consisting of all functions $u \in L_2(0,T;X)$ such that the generalized derivative $u_t = u'$ exists and belongs to $L_2(0,T;Y)$. The norm in $H^1(0,T;X,Y)$ is defined by

$$\|u\|_{H^1(0,T;X,Y)} = (\|u\|^2_{L_2(0,T;X)} + \|u_t\|^2_{L_2(0,T;Y)})^{\frac{1}{2}}.$$

For shortness we set

$$V^{l,0}_{2,\gamma}(Q_T) = L_2(0,T;V^l_{2,\gamma}(G)),$$
$$H^{l,0}_\gamma(Q_T) = L_2(0,T;H^l_\gamma(G)),$$
$$H^{l,0}(Q_T) = L_2(0,T;H^l(G)),$$
$$H^{l,1}(Q_T) = H^1(0,T;H^l(G),L_2(G)),$$
$$H^{-m,0}(Q_T) = L_2(0,T;H^{-m}(G)),$$
$$H^{-m,1}(Q_T) = H^1(0,T;H^{-m}(G),L_2(G)),$$

and

$$\mathcal{H}^{m,1}(Q_T) = H^1(0,T;H^m(G),H^{-m}(G)).$$

Finally, we denote $H^{2ml,l}_\gamma(Q_T)$ $(\gamma \in \mathbb{R})$ the weighted Sobolev space of functions defined in Q equipped with the norm

$$\|u\|_{H^{2ml,l}_\gamma(Q_T)} = \left(\int_{Q_T} (r^{2\gamma} \sum_{|\alpha|+2mk\le 2ml} |D^\alpha u_{t^k}|^2 + \sum_{k=0}^{l} |u_{t^k}|^2) dxdt\right)^{\frac{1}{2}},$$

where $r = |x| = (\sum_{k=1}^n x_k^2)^{\frac{1}{2}}, u_{t^k} = \partial^k u / \partial t^k$.

By $H^{-m}(G)$ we denote the dual space to $H^m(G)$. We write $\langle .,. \rangle$ to denote the pairing between

$H^m(G)$ and $H^{-m}(G)$, and $(.,.)$ to denote the inner product in $L_2(G)$. By identifying $L_2(G)$ with its dual, we have the continuous imbeddings $H^m(G) \hookrightarrow L_2(G) \hookrightarrow H^{-m}(G)$ with the equation

$\left\langle f, \bar{v} \right\rangle = (f, v)$ for $f \in L_2(G) \subset H^{-m}(G), v \in H^m(G)$.

In this paper we consider the following problem

$$u_t + Lu = f \text{ in } Q, \tag{2.5}$$

$$B_j u = 0, \text{ on } S, j = 1, \dots, m, \tag{2.6}$$

$$u = \varphi \text{ on } G, \tag{2.7}$$

where $f : Q \to \mathbb{C}, \varphi : G \to \mathbb{C}$ are given functions.

Let $f \in H^{-m,0}(Q), \varphi \in L_2(G)$. A function $u \in \mathcal{H}^{m,1}(Q)$ is called a generalized solution of the problem (2.5)—(2.7) iff $u(.,0) = \varphi$ and the equality

$$\left\langle u_t, \bar{v} \right\rangle + B(t, u, v) = \left\langle f(t), \bar{v} \right\rangle \tag{2.8}$$

holds for a.e. $t \in (0, +\infty)$ and all $v \in H^m(G)$.

3. SOLVABILITY AND REGULARITY WITH RESPECT TO TIME VARIABLE

In this section we establish the unique existence and the regularity with respect to variable t of generalized solutions of the problem (2.5)—(2.7).

First we introduce the definition of the compatibility conditions imposing on functions f, φ in (2.5) and (2.7). This conditions consist in the fact that the derivatives $u_{t^k} \mid_{t=0}$, which can be determined for $t = 0$ by means of equation (2.5) and the initial condition (2.7), must satisfy for $x \in \Gamma$ the boundary conditions (2.6).

Let $\varphi \in H_\gamma^{(2h+1)m}(G), f \in H_\gamma^{2hm,h}(Q)$, where h is a positive integer, $\gamma \le m$. We set

$$\varphi_0 = \varphi, \varphi_1 = f(.,0) - L(x,0,D)\varphi_0, \dots, \varphi_h =$$
$$= f_{t^{h-1}}(.,0) - \sum_{k=0}^{h-1} \binom{h-1}{k} L_{t^{h-1-k}}(x,0,D)\varphi_k, \tag{3.1}$$

where

$$L_{t^k} = L_{t^k}(x,t,D) = \sum_{|\alpha|,|\beta|=0}^{m} D^\alpha \left(\frac{\partial^k a_{\alpha\beta(x,t)}}{\partial t^k} D^\beta\right).$$

We say that the h^{th}-order compatibility conditions are fulfilled if

$$\sum_{k=0}^{s} \binom{s}{k} (B_j)_{t^{s-k}}(x,0,D)\varphi_k \mid_\Gamma = 0, \tag{3.2}$$
$$s = 0, \dots, h-1,\ j = 1, \dots, m.$$

Now let us state the main theorem of the present section:

Theorem 1. *Let h be a nonnegative integer and γ be a real number, $\gamma \le m$. Assume that $\varphi \in H_\gamma^{(2h+1)m}(G), f \in H_\gamma^{2hm,h}(Q)$ such that $\varphi_0, \dots, \varphi_h \in$ $\in H^m(G)$ and h^{th}-order compatibility conditions are fulfilled in the case $h \ge 1$. Then the problem (2.5)—(2.7) has a unique generalized solution $u \in \mathcal{H}^{m,1}(Q)$, moreover,*

$$u_{t^k} \in H^{m,1}(Q) \text{ for } k = 0, \dots, h, \tag{3.3}$$

and

$$\sum_{k=0}^{h} \left\| u_{t^k} \right\|^2_{H^{m,1}(Q)} \le C\left(\|f\|^2_{H_\gamma^{2hm,h}(Q)} + \sum_{k=0}^{h} \|\varphi_k\|^2_{H^m(G)} \right), \tag{3.4}$$

where C is the constant independent of u, f, φ.

Remark. It follows from $f \in H_\gamma^{2hm,h}(Q)$ that $f_{t^k}(.,0)$ defined in the trace sense and $f_{t^k}(.,0) \in H_\gamma^{(2h-2k-1)m}(G), k = 0, \dots, h-1$. Then it follows from the assumption $\varphi \in H_\gamma^{(2h+1)m}(G)$, $f \in H_\gamma^{2hm,h}(Q)$ that $\varphi_k \in H_\gamma^{(2h-2k+1)m}(G), k = 0, \dots, h$. Thus, by (2.3), $\varphi_k \in H_\gamma^{3m}(G) \subset H_m^{3m}(G) \subset H^m(G)$, $k = 0, \dots, h-1$. However, in general, $\varphi_h \notin H^m(G)$. Therefore, in Theorem 1 instead of the assumption $\varphi_0, \dots, \varphi_h \in H^m(G)$ we can assume only that $\varphi_h \in H^m(G)$.

To prove Theorem 1, we first establish some lemmas.

For simplicity in the following we will some time write $v(t)$ instead of $v(.,t)$ for functions $v(x,t)$ defined on Q. For integer $k \ge 0, u, v \in H^{m,0}(Q_T)$, $t \in [0, +\infty)$ we set

$$B_{t^k}(t,u,v) = \sum_{|\alpha|,|\beta| \le m} \int_G \frac{\partial^k a_{\alpha\beta}(x,t)}{\partial t^k} D^\beta u(x,t) \overline{D^\alpha v}(x,t) dx,$$

$$B_{t^k}^T(u,v) = \int_0^T B_{t^k}(t,u,v)dt, \quad B^T(u,v) = B_{t^0}^T(u,v).$$

Lemma 3.1. *Let $F(t,.,.)$ be a bilinear form on $H^m(G) \times H^m(G)$ satisfying*

$$|F(t,v,w)| \le C\|v\|_{H^m(G)} \|w\|_{H^m(G)} \ (C = \text{const}) \tag{3.5}$$

for all $t \in [0, +\infty)$ and all $v, w \in H^m(G)$, and $F(.,v,w)$ is measurable on $[0,+\infty)$ for each pair $v, w \in H^m(G)$. Assume that $u \in \mathcal{H}^{m,1}(Q)$ satisfies $u(0) \equiv 0$ and

$$\left\langle u_t(t), \bar{v} \right\rangle + B(t, u(t), v) = \int_0^t F(\tau, u(\tau), \bar{v}) d\tau \tag{3.6}$$

for a.e. $t \in [0, +\infty)$ and all $v \in H^m(G)$. Then $u \equiv 0$ on Q.

Proof. Substituting $v := u(t)$ into (3.6), then integrating both sides of the obtained equality with respect to t from 0 to $b (b > 0)$, after all using the assumptions (2.2), (3.5), we arrive at

$$\frac{1}{2}\|u(b)\|^2_{L_2(G)} + \mu\|u\|^2_{H^{m,0}(Q_b)} \le$$
$$\le C\int_0^b\int_0^t \|u(t)\|_{H^m(G)}\|u(\tau)\|_{H^m(G)}\,d\tau dt \le$$
$$\le \frac{1}{2}C\int_0^b\int_0^t (\|u(t)\|^2_{H^m(G)} + \|u(\tau)\|^2_{H^m(G)})d\tau dt \le$$
$$\le bC\|u\|^2_{H^{m,0}(Q_b)}.$$

Choosing $b = \frac{\mu}{2C}$, we have $\frac{1}{2}(\|u(b)\|^2_{L_2(G)} + + \mu\|u\|^2_{H^{m,0}(Q_b)}) \le 0$. This implies $u \equiv 0$ on $[0, \frac{\mu}{2C}]$. Repeating this argument we can show that $u \equiv 0$ on intervals $[\frac{\mu}{2C}, \frac{\mu}{C}], [\frac{\mu}{C}, \frac{3\mu}{2C}], \ldots$, and, therefore, $u \equiv 0$ on Q. □

Lemma 3.2. *If $f \in H^{-m,0}(Q), \varphi \in L_2(G)$, then there exists a unique generalized solution $u \in H^{m,1}(Q)$ of the problem (2.5)—(2.7).*

Proof. The uniqueness of the solution follows directly from Lemma 3.1. We will prove its existence. Since $H^m(G)$ is compact imbedded in $L_2(G)$, we can take a system of functions $\{\psi_k\}_{k=1}^\infty$ which is not only an orthogonal basis of $H^m(G)$ but also an orthonormal basis of $L_2(G)$ (For instance, $\{\psi_k\}_{k=1}^\infty$ is a completed set of normalized eigenfunctions of a formally self-adjoint strong elliptic operator of order $2m$ in G). For each positive integer N, we consider the function $u^N(x,t) = \sum_{k=1}^N C_k^N(t)\psi_k(x)$, where $\{C_k^N(t)\}_{k=1}^N$ is the solution of the ordinary differential system:

$$(u_t^N, \psi_l) + B(t, u^N, \psi_l) = \langle f, \overline{\psi_l}\rangle,\ l = 1,\ldots,N, \quad (3.7)$$
$$C_k^N(0) = C_k, \quad k = 1,\ldots,N. \quad (3.8)$$

Here $C_k = (\varphi, \psi_k), k = 1,2,\ldots$. After multiplying both sides of (3.7) by $\overline{C_l^N}(t)$, taking sum with respect to l from 1 to N, and integrating with respect to t from 0 to T $(T > 0)$, we arrive at

$$\int_0^T (u_t^N, u^N)dt + B^T(u^N, u^N) = \int_0^T \langle f, \overline{u^N}\rangle dt. \quad (3.9)$$

Adding (3.9) with its complex conjugate, we obtain

$$\|u^N(T)\|^2_{L_2(G)} + 2B^T(u^N, u^N) =$$
$$= \|u^N(0)\|^2_{L_2(G)} + 2\mathrm{Re}\int_0^T \langle f, \overline{u^N}\rangle dt. \quad (3.10)$$

Noting that $\|u^N(0)\|^2_{L_2(G)} = \left\|\sum_{k=1}^N (\varphi, \psi_k)\psi_k\right\|^2_{L_2(G)} \le$ $\le \|\varphi\|^2_{L_2(G)}$ and

$$\left|2\mathrm{Re}\int_0^T \langle f, \overline{u^N}\rangle dt\right| \le 2\int_0^T \|f\|_{H^{-m}(G)}\|u^N\|_{H^m(G)}\,dt \le$$
$$\le \varepsilon\|u^N\|^2_{H^{m,0}(Q_T)} + \frac{1}{\varepsilon}\|f\|^2_{H^{-m,0}(Q_T)}$$

$(0 < \varepsilon < 2\mu)$, and using the assumption (2), we have from (19) that

$$\|u^N\|^2_{H^{m,0}(Q_T)} \le C(\|\varphi\|^2_{L_2(G)} + \|f\|^2_{H^{-m,0}(Q_T)}).$$

Sending $T \to +\infty$, we obtain

$$\|u^N\|^2_{H^{m,0}(Q)} \le C(\|\varphi\|^2_{L_2(G)} + \|f\|^2_{H^{-m,0}(Q)}). \quad (3.11)$$

Now fix any $v \in H^m(G)$ with $\|v\|_{H^m(G)} \le 1$, and write $v = v_1 + v_2$, where $v_1 \in \mathrm{span}\{\psi_l\}_{l=1}^N, (v_2, \psi_l)_{L_2(G)} = = 0, l = 1,\ldots,N$. Since the functions $\{\psi_l\}_{l=1}^N$ are orthogonal in $H^m(G), \|v_1\|_{H^m(G)} \le \|v\|_{H^m(G)} \le 1$. We obtain from (3.7) that

$$(u_t^N, v_1) + B(t, u^N, v_1) = \langle f, \overline{v_1}\rangle.$$

Therefore,

$$\langle u_t^N, v\rangle = (u_t^N, v) = (u_t^N, v_1) = \langle f, \overline{v_1}\rangle - B(t, u^N, v_1).$$

Hence, we get

$$\left|\langle u_t^N, v\rangle\right| \le C\left(\|f\|_{H^{-m}(G)} + \|u^N\|_{H^m(G)}\right)$$

since $\|v_1\|_{H^m(G)} \le 1$. Thus,

$$\|u_t^N\|_{H^{-m}(G)} \le C\left(\|f\|_{H^{-m}(G)} + \|u^N\|_{H^m(G)}\right),$$

and therefore,

$$\|u_t^N\|^2_{H^{-m,0}(Q)} \le C\left(\|f\|^2_{H^{-m,0}(Q)} + \|u^N\|^2_{H^{m,0}(Q)}\right) \le$$
$$\le C(\|\varphi\|^2_{L_2(G)} + \|f\|^2_{H^{-m,0}(Q)}). \quad (3.12)$$

Combining (3.11) and (3.12), we get

$$\|u^N\|^2_{H^{m,1}(Q)} \le C\left(\|\varphi\|^2_{L_2(G)} + \|f\|^2_{H^{-m,0}(Q)}\right), \quad (3.13)$$

where C is a constant independent of φ, f and N. From this estimate, by the same arguments as in [1. Ch. 7. Th. 3], we conclude that there exists a subsequence of $\{u^N\}$ which weakly converges to a generalized solution $u \in \mathcal{H}^{m,1}(Q)$ of the problem (2.5)—(2.9). □

Lemma 3.3. *Let $\varphi \in H^m(G)$ and $f \in L_2(Q)$ or $f \in H^{-m,1}(Q)$. Then the generalized solution $u \in \mathcal{H}^{m,1}(Q)$ of the problem (2.5)—(2.9) in fact belongs to $H^{m,1}(Q)$ and the following estimate*

$$\|u\|^2_{H^{m,1}(Q)} \le C\left(\|\varphi\|^2_{H^m(G)} + \|f\|^2_X\right) \quad (3.14)$$

holds with the constant C independent of g, f, and u. Here X is $L_2(Q)$ or $H^{-m,1}(Q)$ according as $f \in L_2(Q)$ or $f \in H^{-m,1}(Q)$.

Proof. (i) Let us consider first the case $f \in L_2(Q)$. Let u^N be the functions defined as in the proof of Theorem 3.2 with $C_k = (\varphi, \psi_k)$ $(k = 1, 2, \dots)$ replaced by $C_k = \|\psi_k\|^{-2}_{H^m(G)} (\varphi, \psi_k)_{H^m(G)}$, where $(.,.)_{H^m(G)}$ denotes the inner product in $H^m(G)$. Multiplying both sides of (3.7) by $\frac{d\overline{C_l^N}}{dt}$, then taking sum with respect to l from 1 to N, after that integrating with respect to t from 0 to T $(0 < T < +\infty)$, and adding the attained equality with its complex conjugate, we arrive at

$$2\left\|u_t^N\right\|^2_{L_2(Q_T)} + \sum_{|\alpha|,|\beta|=0}^{m} \int_{Q_T} a_{\alpha\beta} \frac{\partial}{\partial t}(D^\beta u^N \overline{D^\alpha u^N}) dxdt =$$
$$= 2\mathrm{Re}\int_0^T (f, u_t^N) dt.$$

By the integration by parts, we get

$$2\left\|u_t^N\right\|^2_{L_2(Q_T)} + B(T, u^N, u^N) =$$
$$= B(0, u^N, u^N) + B_t^T(u^N, u^N) + 2\mathrm{Re}\int_0^T (f, u_t^N) dt. \quad (3.15)$$

Since $a_{\alpha\beta}, \frac{\partial a_{\alpha\beta}}{\partial t}$ are bounded on $\overline{Q}$, using Cauchy's inequality, we get

$$\left|B(0, u^N, u^N)\right| \le C\left\|u^N(0)\right\|^2_{H^m(G)} \le C\left\|\varphi\right\|^2_{H^m(G)},$$
$$\left|B_t^T(u^N, u^N)\right| \le C\left\|u^N\right\|^2_{H^{m,0}(Q_T)},$$
$$\left|2\mathrm{Re}\int_0^T (f, u_t^N) dt\right| \le$$
$$\le \varepsilon\left\|u_t^N\right\|^2_{L_2(Q_T)} + \frac{1}{4\varepsilon}\left\|f\right\|^2_{L_2(Q_T)} \ (0 < \varepsilon < 2).$$

Hence, it follows from (3.11) and (3.15) that

$$\left\|u_t^N\right\|^2_{L_2(Q_T)} \le C(\left\|\varphi\right\|^2_{H^m(G)} + \left\|f\right\|^2_{L_2(Q_T)}). \quad (3.16)$$

Sending $T \to +\infty$, we obtain

$$\left\|u_t^N\right\|^2_{L_2(Q)} \le (\left\|\varphi\right\|^2_{H^m(G)} + \left\|f\right\|^2_{L_2(Q)}). \quad (3.17)$$

Combining (3.11) and (3.17), we have

$$\left\|u^N\right\|^2_{H^{m,1}(Q)} \le C(\left\|\varphi\right\|^2_{H^m(G)} + \left\|f\right\|^2_{L_2(Q)}).$$

This implies that the sequence $\{u^N\}$ contains a subsequence which weakly converges to a function $v \in H^{m,1}(Q)$. Passing to the limit of the subsequence, we can see that v is a generalized solution of the problem (2.5)—(2.7). Thus, $u = v \in H^{m,1}(Q)$. The estimate (3.14) with $X = L_2(Q)$ follows from (3.17).

(ii) Now let $f \in H^{-m,1}(Q)$. Then f, as a function from $[0, +\infty)$ to $H^{-m}(G)$, is continuous on $[0, +\infty)$ and has the representation $f(t) = f(s) + \int_s^t f_t(\tau)d\tau$ for all $s, t \in [0, +\infty)$ (see [1, Sec. 5.9, Th. 2]). This implies

$$\left\|f(t)\right\|^2_{H^{-m}(G)} \le 2\left\|f(s)\right\|^2_{H^{-m}(G)} + 2\int_J \left\|f_t(\tau)\right\|^2_{H^{-m}(G)} d\tau, \quad (3.18)$$

where $J = [a, b] \subset [0, +\infty)$ such that $a \le t \le b$ and $b - a = 1$. Integrating both sides of (3.18) with respect to s on J, we obtain

$$\left\|f(t)\right\|^2_{H^{-m}(G)} \le 2\left\|f\right\|^2_{H^{-m,1}(Q)} \ (t \in [0, +\infty)). \quad (3.19)$$

Now by the same way to get (3.15), we have

$$2\left\|u_t^N\right\|_{L_2(Q_T)} + B(T, u^N, u^N) =$$
$$= B(0, u^N, u^N) + B_t^T(u^N, u^N) + 2\mathrm{Re}\int_0^T \left\|f, u_t^N\right\| dt. \quad (3.20)$$

Noting that $\int_0^T \left\langle f, u_t^N \right\rangle dt = -\int_0^T \left\langle f_t, u^N \right\rangle dt + \left.\left\langle f, u^N \right\rangle\right|_0^T$, and using (3.19), we obtain

$$\left|\int_0^T \left\langle f, u_t^N \right\rangle dt\right| \le \left\|f_t\right\|_{H^{-m,0}(Q)} \left\|u^N\right\|_{H^{m,0}(Q)} +$$
$$+ \left\|f(T)\right\|_{H^{-m}(G)} \left\|u^N(T)\right\|_{H^m(G)} +$$
$$+ \left\|f(0)\right\|_{H^{-m}(G)} \left\|u^N(0)\right\|_{H^m(G)} \le$$
$$\le C(\varepsilon)\left\|f\right\|^2_{H^{-m,1}(Q)} + \varepsilon(\left\|u^N\right\|^2_{H^{m,0}(Q_T)} +$$
$$+ \left\|u^N(T)\right\|^2_{H^m(G)} + \left\|u^N(0)\right\|^2_{H^m(G)}). \quad (3.21)$$

Using (3.11), (3.12) and (3.21) for $0 < \varepsilon < \mu$, we get from (3.20) that

$$\left\|u_t^N\right\|^2_{L_2(Q_T)} \le C(\left\|\varphi\right\|^2_{H^m(G)} + \left\|f\right\|^2_{H^{-m,1}(Q)}).$$

Sending $T \to +\infty$, we can see

$$\left\|u_t^N\right\|^2_{L_2(Q)} \le C(\left\|\varphi\right\|^2_{H^m(G)} + \left\|f\right\|^2_{H^{-m,1}(Q)}). \quad (3.22)$$

Combining (3.11) and (3.22), we have

$$\left\|u^N\right\|^2_{H^{m,1}(Q)} \le C(\left\|\varphi\right\|^2_{H^m(G)} + \left\|f\right\|^2_{H^{-m,1}(Q)}). \quad (35)$$

From this, by the same argument as in the part (i) above, we obtain the assertion of the lemma for the case $f \in H^{-m,1}(Q)$. □

Remark. It follows from the proof of Lemma 3.3 that if $\varphi \in H^m(G)$ and $f = f_1 + f_2$, where $f_1 \in L_2(Q), f_2 \in H^{-m,1}(Q)$ then the generalized solution $u \in \mathcal{H}^{m,1}(Q)$ of the problem (2.5)—(2.7) belongs to $H^{m,1}(Q)$ and the estimate (3.14) holds with $\left\|f\right\|^2_X$ replaced by $\left\|f_1\right\|^2_{L_2(Q)} + \left\|f_2\right\|^2_{H^{-m,1}(Q)}$.

Proof of Theorem 1: We will show by induction on h that not only the assertions (3.3), (3.4) but also the following equalities hold:

$$u_{t^k}(0) = \varphi_k, k = 1, \dots, h, \quad (3.24)$$

and

$$(u_{t^{h+1}},\eta)+\sum_{k=0}^{h}\binom{h}{k}B_{t^{h-k}}(t,u_{t^k},\eta)=(f_{t^h},\eta) \quad (3.25)$$

for all $\eta\in H^m(G)$.

The case $h=0$ follows from Lemmas 3.2, 3.3. Assuming now that they hold for $h-1$, we will prove them for $h (h\geq 1)$. We consider first the following problem: find a function $v\in\mathcal{H}^{m,1}(Q)$ satisfying $v(0)=i_h$ and

$$\langle v_t,\bar{\eta}\rangle+B(t,v,\eta)=$$
$$=(f_{t^h},\eta)-\sum_{k=0}^{h-1}\binom{h}{k}B_{t^{h-k}}(t,u_{t^k},\eta) \quad (3.26)$$

for all $\eta\in H^m(G)$ and a.e. $t\in(0,+\infty)$.

Let $F(t), t\in[0,+\infty)$, be functionals defined by

$$\langle F(t),\bar{\eta}\rangle=(f_{t^h},\eta)-\sum_{k=0}^{h-1}\binom{h}{k}B_{t^{h-k}}(t,u_{t^k},\eta),$$
$$\eta\in H^m(G). \quad (3.27)$$

Then $F\in H^{-m,0}(Q)$ by the induction hypothesis. Hence, according to Lemma 3.2, the problem (38) has a solution $v\in\mathcal{H}^{m,1}(Q)$. We put now

$$w(x,t)=\varphi_{h-1}(x)+\int_0^t v(x,\tau)d\tau,\ x\in G, t\in[0,+\infty).$$

Then we have $w(0)=\varphi_{h-1}, w_t=v, w_t(0)=\varphi_h$. It follows from (3.26) that

$$\langle w_{tt},\bar{\eta}\rangle+\frac{\partial}{\partial t}B(t,w,\eta)=$$
$$=(f_{t^h},\eta)+B_t(t,w-u_{t^{h-1}},\eta)-$$
$$-\frac{\partial}{\partial t}\sum_{k=0}^{h-2}\binom{h-1}{k}B_{t^{h-1-k}}(t,u_{t^k},\eta). \quad (3.28)$$

It follows from equality (2.1) that

$$\int_G L\psi\bar{\eta}dx=B(t,\psi,\eta)+\sum_{j=1}^{m}\int_\Gamma B_j\psi\overline{B'_j\eta}ds$$

for $\psi\in H^{2m}_\gamma(G),\eta\in H^m(G)$ and all $t\in[0,+\infty)$. Differentiating both sides of this equality with respect to t $h-1-k$ times and taking $\psi=\varphi_k$ $(0\leq k\leq h-1)$, we have

$$\int_G L_{t^{h-1-k}}\varphi_k\bar{\eta}dx=B_{t^{h-1-k}}(t,\varphi_k,\eta)+$$
$$+\sum_{j=1}^{m}\int_\Gamma\sum_{s=0}^{h-1-k}\binom{h-1-k}{s}(B_j)_{t^{h-1-k-s}}\varphi_k\overline{(B'_j)_{t^s}\eta}ds.$$

Multiplying both sides of this equality with $\binom{h-1}{k}$, taking sum in k from 0 to $h-1$ and noting that $\binom{h-1}{k}\binom{h-1-k}{s}=\binom{h-1}{s}\binom{h-1-s}{k}$, we have

$$\int_G\sum_{k=0}^{h-1}\binom{h-1}{k}L_{t^{h-1-k}}\varphi_k\bar{\eta}dx=$$
$$=\sum_{k=0}^{h-1}\binom{h-1}{k}B_{t^{h-1-k}}(t,\varphi_k,\eta)+$$
$$+\sum_{j=1}^{m}\sum_{s=0}^{h-1}\binom{h-1}{s}\int_\Gamma\sum_{k=0}^{h-1-s}\binom{h-1-s}{k}(B_j)_{t^{h-1-s-k}}\varphi_k\overline{(B'_j)_{t^s}\eta}ds. \quad (3.29)$$

From this equality taking $t=0$ together with (3.1) and (3.2) we obtain

$$(\varphi_h,\eta)=(f_{t^{h-1}}(0),\eta)-\sum_{k=0}^{h-1}\binom{h-1}{k}B_{t^{h-1-k}}(0,\varphi_k,\eta). \quad (3.30)$$

Now integrating equality (3.28) with respect to t from 0 to t and using (3.30), we arrive at

$$\langle w_t,\bar{\eta}\rangle+B(t,w,\eta)=(f_{t^{h-1}},\eta)+$$
$$+\int_0^t B_t(\tau,w-u_{t^{h-1}},\eta)d\tau-\sum_{k=0}^{h-1}\binom{h-1}{k}B_{t^{h-1-k}}(t,u_{t^k},\eta). \quad (3.31)$$

Put $z=w-u_{t^{h-1}}$. Then $z(0)=0$ since $u(0)==w(0)=\varphi_{h-1}$. It follows from the induction hypothesis (3.25) with h replaced by $h-1$ and (3.31) that

$$\langle z_t(t),\bar{\eta}\rangle+B(t,z(t),\eta)=\int_0^t B_t(\tau,z(.\tau),\eta)d\tau. \quad (3.32)$$

Applying Lemma 3.1, we can see from (3.32) that $z\equiv 0$ on Q. Therefore, $u_{t^h}=w_t=v\in\mathcal{H}^{m,1}(Q)$.

Now we show that in fact $u_{t^h}\in\mathcal{H}^{m,1}(Q)$. We rewrite (3.26) in the form

$$\langle v_t,\bar{\eta}\rangle+B(t,v,\eta)=(f_{t^h},\eta)+\langle\widehat{F}(t),\eta\rangle, \quad (3.33)$$

where $\widehat{F}(t), t\in[0,+\infty)$, are functionals on $H^m(G)$ defined by

$$\langle\widehat{F}(t),\bar{\eta}\rangle=-\sum_{k=0}^{h-1}\binom{h}{k}B_{t^{h-k}}(t,u_{t^k},\eta),\ \eta\in H^m(G). \quad (3.34)$$

Since $u_{t^k}\in H^{m,0}(Q)$ for $k=0,\dots,h$, we can see from (3.34) that $\widehat{F}_t\in H^{-m,0}(G)$ and

$$\langle\widehat{F}_t(t),\bar{\eta}\rangle=-\sum_{k=0}^{h-1}\binom{h+1}{k}B_{t^{h+1-k}}(t,u_{t^k},\eta)-$$
$$-hB_t(t,u_{t^h},\eta),\ \eta\in H^m(G).$$

Then, according to the remark below Lemma 3.3, we obtain from (3.33) that $u_{t^h}=v\in H^{m,1}(Q)$. The inequality (3.4) holds since $\|f_{t^h}\|_{L_2(Q)}$ and $\|\widehat{F}_t\|_{H^{m,0}(Q)}$ can be estimated by the right-hand side of it. The proof is completed.

4. GLOBAL REGULARITY OF THE SOLUTIONS

Let $L_0(x,t,D)$, $B_{0j}(x,t,D)$ be the principal homogeneous parts of $L(x,t,D)$, $B_j(x,t,D)$. We can write $L_0(0,t,D), B_{0j}(0,t,D)$ in the form

$$L_0(0,t,D) = r^{-2m}\mathcal{L}(\omega,t,D_\omega,rD_r), \qquad (4.1)$$

$$B_{0,j}(0,t,D) = r^{-m_j}\mathcal{B}_j(\omega,t,D_\omega,rD_r), \qquad (4.2)$$

where $r = |x|$, ω is an arbitrary local coordinate system on S^{n-1}, $D_r = -i\partial/\partial r$. We denote by $\mathcal{U}(\lambda,t) = (\mathcal{L}(\omega,t,D_\omega,\lambda), \mathcal{B}_j(\omega,t,D_\omega,\lambda))$ $(\lambda \in \mathbb{C}, t \in (0,+\infty))$ the operator of the parameter-depending boundary problem

$$L(\omega,t,D_\omega,\lambda) = f \text{ in } \Omega, \qquad (4.3)$$

$$B_j(\omega,t,D_\omega,\lambda) = g_j \text{ on } \partial\Omega, j = 1,\dots,m. \quad (4.4)$$

For every fixed $\lambda \in \mathbb{C}$ this operator continuously maps

$$H^l(\Omega) \text{ into } H^{l-2m}(\Omega) \times \prod_{j=1}^{m} H^{l-m_j-\frac{1}{2}}(\partial\Omega)\ (l \geq 2m).$$

For each $t \in (0,+\infty)$ we have the operator pencil $\mathcal{U}(\lambda,t)$ which has the spectrum being an enumerable set of eigenvalues (see [5, Th. 5.2.1]).

Now we first give the main theorem of this section:

Theorem 2. *Suppose that the assumptions of Theorem 1 hold. Assume further that* $0 \leq \gamma \leq m$, $0 \leq \gamma \leq m, \gamma + \frac{n}{2} \notin \{1,\dots,2(h+1)m\}$ *and the strip* $\gamma - 2hm - 2m + \frac{n}{2} \leq \operatorname{Im}\lambda \leq -m + \varepsilon + \frac{n}{2}$ *does not contain any eigenvalue of* $\mathcal{U}(\lambda,t)$ *for all* $t \in (0,+\infty)$, *where* ε *is some nonnegative number such that* $\varepsilon + \frac{n}{2} \notin \{1,\dots,m\}$. *Then* $u \in H_\gamma^{(2h+2)m,h+1}(Q)$ *and*

$$\|u\|^2_{H_\gamma^{(2h+2)m,h+1}(Q)} \leq C\left(\|f\|^2_{H_\gamma^{2hm,h}(Q)} + \sum_{k=0}^{h}\|\varphi_k\|^2_{H^m(G)}\right), \quad (4.5)$$

where C *is the constant independent of* u, f, φ.

Before we prove Theorem 2 we will need some lemmas. The following lemma can be proved similarly to Lemma 3 of [4], Theorems 4.2, 4.2' of [9].

Lemma 1. *For every fixed* $t_0 \in [0,+\infty)$ *let* $u \in H_{\text{loc}}^{l+2m}(\overline{G}\setminus\{0\}) \cap V^0_{2,\gamma-l-2m}(G)$ *be a solution of the problem*

$$L(x,t_0,D)u = f \text{ in } G \qquad (4.6)$$

$$B_j(x,t_0,D)u = g_j \text{ on } \Gamma, j = 1,\dots,m, \quad (4.7)$$

where $f \in V^l_{2,\gamma}(G), g_j \in V_{2,\gamma}^{l+2m-m_j-\frac{1}{2}}(\Gamma), l$ is a nonnegative integer. Then $u \in V_{2,\gamma}^{l+2m}(G)$ and the following estimate

$$\|u\|^2_{V_{2,\gamma}^{l+2m}(G)} \leq$$

$$\leq C\left(\|f\|^2_{V^l_{2,\gamma}(G)} + \sum_{j=1}^{m}\|g_j\|^2_{V_{2,\gamma}^{l+2m-m_j-\frac{1}{2}}(\Gamma)} + \|u\|^2_{V^0_{2,\gamma-l-2m}(G)}\right) \quad (4.8)$$

holds with the constant C *independent of* u, f, g_j *and* t_0.

Let ε is an arbitrary positive number. We introduce the following integral operator

$$(Kw)(r) = \xi(r)\int_{\frac{1}{2}}^{1} w(tr)\psi(t)dt \quad \text{for } 0 < r < \varepsilon \quad (4.9)$$

where ξ is a cut-off function on $[0,+\infty)$ equal to one in $\left[0,\frac{\varepsilon}{2}\right)$ and to zero outside $[0,\varepsilon)$, and $\psi \in C_0^\infty\left(\left(\frac{1}{2},1\right)\right)$ satisfying the condition $\int_{\frac{1}{2}}^{1}\psi(t)dt = 1$. For $r > \varepsilon$ we set $(Kw)(r) = 0$.

It is known (see [5, Le. 7.3.3]) that K is a continuous mapping

$$H^{\frac{1}{2}}((0,\varepsilon)) \to H^l_{l-\frac{1}{2}}((0,+\infty)) \qquad (4.10)$$

for arbitrary integer $l \geq 1$, where $H^{\frac{1}{2}}((0,\varepsilon))$ is the space of all functions defined on $(0,\varepsilon)$ with the finite norm

$$\|u\|_{H^{\frac{1}{2}}((0,\varepsilon))} = \left(\|u\|^2_{L_2((0,\varepsilon))} + \int_0^\varepsilon\int_0^\varepsilon\left|\frac{u(r)-u(\rho)}{r-\rho}\right|^2 drd\rho\right)^{\frac{1}{2}}.$$

In the following, by $p_k(u)$ we mean the Taylor polynomial at the point $x = 0$ of degree k of the function u defined in G if it exists.

Lemma 4.2. *Let* $u \in H^l_\gamma(G)$, *where* $0 < \gamma + \frac{n}{2} \leq l$. *Then for an arbitrary integer* $k \geq 0$, u *admits the representation* $u = v + w$, *where* $v \in V^l_{2,\gamma}(G)$ *and* $w \in H^{l+k}_{\gamma+k}(G)$, *moreover,*

$$\|v\|^2_{V^l_{2,\gamma}(G)} + \|w\|^2_{H^{l+k}_{\gamma+k}(G)} \leq C\|u\|^2_{H^l_\gamma(G)} \qquad (4.11)$$

with the constant C *independent of* u.

Proof. Let $s = [\gamma + \frac{n}{2}]$ be the greatest integer not exceeding $\gamma + \frac{n}{2}$. Denote by ζ a smooth function equal to one near the origin and to zero outside a neighborhood in which G coincides with the cone K. By the [5, Th. 7.3.1, 7.3.2], $u \in H^l_\gamma(G)$ can be written in the form $u = v + w$, where $v \in V^l_{2,\gamma}(G)$ with the norm estimated by $\|u\|_{H^l_\gamma(G)}$, and

$$w = \zeta p_{l-s-1}(u)$$

if $\gamma + \frac{n}{2}$ is not integer, or

$$w = \zeta p_{l-s-1}(u) + \zeta \sum_{|\alpha|=l-s} (Ku_\alpha)(|x|)x^\alpha \alpha!$$

if $\gamma + \frac{n}{2}$ is integer, the coefficients of $p_{l-s-1}(u)$ are estimated by $\|u\|_{H^l_\gamma(G)}$, u_α are functions from $H^{\frac{1}{2}}((0,\varepsilon))$. It is obvious that $\zeta p_{l-s-1}(u) \in H^{l+k}_{\gamma+k}(G)$ and

$$\|\zeta p_{l-s-1}(u)\|^2_{H^{l+k}_{\gamma+k}(G)} \le C\|u\|^2_{H^l_\gamma(G)} \ (C = \text{const}).$$

By (4.10) we have $(Ku_\alpha)(r) \in H^{l+k}_{l+k-\frac{1}{2}}((0,+\infty))$. Hence, see [5, Le. 7.3.1], $\zeta(Ku_\alpha)(|x|) \in H^{l+k}_{l-\frac{n}{2}+k}(G) \equiv$ $\equiv H^{l+k}_{l-s+\gamma+k}(G)$ ($l - s = l - \frac{n}{2} - \gamma$ for $\gamma + \frac{n}{2}$ is integer). Thus, $\zeta \sum_{|\alpha|=l-s} (Ku_\alpha)(|x|)\frac{x^\alpha}{\alpha!} \in H^{l+k}_{\gamma+k}(G)$ with the norm estimated by $\|u\|_{H^l_\gamma(G)}$. The lemma is proved. □

Lemma 4.3. *For every fixed $t_0 \in (0,+\infty)$ let $f \in H^0_m(G)$ and $u \in H^m(G)$ be a generalized solution of the problem (4.6), (4.7), i.e u satisfies the identity*

$$B(t_0, u, \eta) = (f, \eta) \text{ for all } \eta \in H^m(G).$$

Then $u \in H^{2m}_m(G)$ and

$$\|u\|^2_{H^{2m}_m(G)} \le C(\|f\|^2_{H^0_m(G)} + \|u\|^2_{H^m(G)}), \quad (4.12)$$

where the constant C is independent of u, f and t_0.

Proof. According to results for elliptic boundary value problem in domains with smooth boundaries, we have $u \in H^{2m}_{\text{loc}}(\overline{G} \setminus \{0\})$. If $m < \frac{n}{2}$, then $H^m(G) = V^m_{2,0}(G)$ by [5, Th. 7.1.1]. Thus the assertion of the lemma follows from Lemma 4.1.

Let us consider the case $m \ge \frac{n}{2}$. According to Lemma 4.2, $u \in H^m(G)$ can be written in the form $u = v + w$, where $v \in V^m_{2,0}(G)$, $w \in H^{2m}_m(G)$, and

$$\|v\|^2_{V^m_{2,0}(G)} + \|w\|^2_{H^{2m}_m(G)} \le C\|u\|^2_{H^m(G)} \quad (C = \text{const}). \ (4.13)$$

Now we rewrite (4.6), (4.7) in the form

$$L(x, t_0, D)v = F \text{ in } G, \quad (4.14)$$

$$B_j(x, t_0, D)v = \psi_j \text{ on } \Gamma, \ j = 1,\dots,m, \quad (4.15)$$

where $F = f - L(x, t_0, D)w \in H^0_m(G) \equiv V^0_{2,m}(G)$, $\psi_j = -B_j(x, t_0, D)w \in H_m^{2m-m_j-\frac{1}{2}}(\Gamma)$. Since $m_j \ge m$ for each $j = 1,\dots,m$, we have $m > 2m - m_j - \frac{1}{2}$, and therefore, by [5. Th. 7.1.1], $H_m^{2m-m_j-\frac{1}{2}}(\Gamma) =$ $= V_{2,m}^{2m-m_j-\frac{1}{2}}(\Gamma)$. Thus, $\psi_j \in V_{2,m}^{2m-m_j-\frac{1}{2}}(\Gamma), j = 1,\dots,m$.

Now applying Lemma 4.1, we can see from (4.14), (4.15) that $v \in V^{2m}_{2,m}(G)$ and

$$\|v\|^2_{V^{2m}_{2,m}(G)} \le C\left(\|F\|^2_{V^0_{2,m}(G)} + \sum_{j=1}^m \|\psi_j\|^2_{V_{2,m}^{2m-m_j-\frac{1}{2}}(\Gamma)} + \|v\|^2_{V^0_{2,-m}(G)}\right) \le$$

$$\le C\left(\|f\|^2_{H^0_m(G)} + \|w\|^2_{H^{2m}_m(G)} + \sum_{j=1}^m \|\psi_j\|^2_{H_m^{2m-m_j-\frac{1}{2}}(\Gamma)} + \|v\|^2_{V^m_{2,0}(G)}\right) \le$$

$$\le C\left(\|f\|^2_{H^0_m(G)} + \sum_{j=1}^m \|\psi_j\|^2_{H_m^{2m-m_j-\frac{1}{2}}(\Gamma)} + \|u\|^2_{H^m(G)}\right).$$

Therefore, $u = v + w \in H^{2m}_m(G)$ and the inequality (4.12) holds.

Lemma 4.4. *Let $u \in H^{l+2m,0}_\gamma(Q)$ be a solution of the problem*

$$L(x, t, D)u = f \text{ in } Q \quad (4.16)$$

$$B_j(x, t, D)u = g_j \text{ on } S, \ j = 1,\dots,m \quad (4.17)$$

where $f \in H^{k,0}_\delta(Q), g_j \in H_\delta^{k+2m-m_j-\frac{1}{2},0}(S), l, k$ are a nonnegative integer, $k - \delta > l - \gamma, \gamma + \frac{n}{2} \notin \{1,\dots,l\}$, $\delta + \frac{n}{2} \notin \{1,\dots,k\}$. Suppose that the strip $\delta - k - 2m + \frac{n}{2} \le \operatorname{Im}\lambda \le \gamma - l - 2m + \frac{n}{2}$ does not contain any eigenvalue of $\mathcal{U}(\lambda, t)$ for all $t \in (0,+\infty)$. Then $u \in H^{k+2m,0}_\delta(Q)$ and

$$\|u\|^2_{H^{k+2m,0}_\delta(Q)} \le C\left(\|f\|^2_{H^{k,0}_\delta(Q)} + \sum_{j=1}^m \|g_j\|^2_{H_\delta^{k+2m-m_j-\frac{1}{2},0}(S)} + \|u\|^2_{H^{l+2m,0}_\gamma(Q)}\right) \quad (4.18)$$

with the constant C independent of u, f, g_j.

Proof. First, we fix $t \in (0,+\infty)$ and consider (4.16), (4.17) as an elliptic boundary value problem. Since coefficients of $L(x,t,D), B_j(x,t,D)$ are bounded smooth functions, we can apply [5, Th. 7.2.4] to conclude from (4.16), (4.17) that $u(t) \in H^{k+2m}_\delta(G)$ and

$$\|u(t)\|_{H^{k+2m}_\delta(G)} \le C\left(\|f(t)\|_{H^k_\delta(G)} + \sum_{j=1}^m \|g_j(t)\|_{H_\delta^{k+2m-m_j-\frac{1}{2}}(\Gamma)} + \|u(t)\|^2_{H^{l+2m}_\gamma(G)}\right), \quad (4.19)$$

where the constant C is independent of u, f, g_j and t. Now integrating both sides of (4.19) with respect to t from 0 to $+\infty$, we get the assertion of the lemma.

Proof of Theorem 2: The proof is an induction on h. Let us consider first the case $h = 0$. We rewrite (2.5), (2.6) in the form

$$Lu = f_1 := f - u_t \text{ in } Q, \quad (4.20)$$

$$B_j u = 0 \text{ on } S,\ j = 1,\dots,m. \quad (4.21)$$

According to Theorem 1, we have $u_t \in L_2(Q)$. Thus, $f_1 \in H_\gamma^{0,0}(Q) \subset H_m^{0,0}$ since $0 \le \gamma \le m$. By Lemma 4.3, it follows from (4.20) that $u(.,t) \in H_m^{2m}(G)$ and

$$\|u(t)\|^2_{H_m^{2m}(G)} \le C\left(\|f_1(t)\|^2_{L_2(G)} + \|u(t)\|^2_{H^m(G)}\right) \quad (4.22)$$

for a.e. $t \in (0,+\infty)$, where C is a constant independent of u, f_1 and t. Integrating both sides of (4.22) with respect to t from 0 to $+\infty$, we obtain $u \in H_m^{2m,0}(Q)$. Since the strip $\gamma - 2m + \frac{n}{2} \le \operatorname{Im}\lambda \le -m + \frac{n}{2}$ is free of eigenvalues of $\mathcal{U}(\lambda,t)$ for all $t \in (0,+\infty)$, we have $u \in H_\gamma^{2m,0}(Q)$ by Lemma 4.4. This and the fact that $u_t \in L_2(Q)$ imply $u \in H_\gamma^{2m,1}(Q)$. Moreover, we can get estimate (4.5) for $h = 0$ from (4.22) and (3.3) with $h = 0$. Hence, the theorem is valid for $h = 0$.

Assume that it is true for some nonnegative $h-1$. Then we have $u \in H_\gamma^{2hm,h}(Q)$. Thus,

$$u_{t^s} \in H_\gamma^{(2h-2s)m,0}(Q), s \le h. \quad (4.23)$$

We prove now the theorem for h. Then we have to show that $u \in H_\gamma^{(2h+2)m,h+1}(Q)$. To this end, it is only needed to make clear that

$$u_{t^k} \in L_2(Q), k \le h+1 \quad (4.24)$$

and

$$u_{t^k} \in H_\gamma^{(2h-2k+2)m,0}(Q) \quad (4.25)$$

for $k \le h+1$. (4.24) is valid according to Theorem 1. We will also prove (4.25) by induction on k. By Theorem 1, $u_{t^{h+1}} \in L_2(Q) \subset H_\gamma^{0,0}(Q)$. This means that (4.25) holds for $k = h+1$. Assume that it holds for $k = h+1, h, \dots, p+1$ $(0 < p < h)$. Differentiating both sides of (4.20), (4.21) with respect to t p times, we have

$$Lu_{t^p} = f_{t^p} - u_{t^{p+1}} - \sum_{s=0}^{p-1}\binom{p}{s} L_{t^{p-s}} u_{t^s} \text{ in } Q, \quad (4.26)$$

$$B_j u_{t^p} = -\sum_{s=0}^{p-1}\binom{p}{s} (B_j)_{t^{p-s}} u_{t^s} \text{ on } S,\ j = 1,\dots,m. \quad (4.27)$$

By (4.23), we have $u_{t^s} \in H_\gamma^{(2h-2s)m,0}(Q) \subset H_\gamma^{(2h-2p+2)m,0}(Q)$, $s \le p-1$. Moreover, $u_{t^{p+1}} \in H_\gamma^{(2h-2p)m,0}(Q)$ by the the induction hypothesis, and $f_{t^p} \in H_\gamma^{(2h-2p)m,0}(Q)$ by the assumption of the theorem. Therefore, the right-hand side of (4.26) belongs to $H_\gamma^{(2h-2p)m,0}(Q)$. It follows from $u_{t^s} \in H_\gamma^{(2h-2p+2)m,0}(Q)$ that $(B_j)_{t^{p-s}} u_{t^s} \in H_\gamma^{(2h-2p+2)m-m_j-\frac{1}{2},0}(S)$. Hence, by Lemma 4.4, $u_{t^p} \in H_\gamma^{(2h-2p+2)m,0}(Q)$. Thus, (4.25) holds for $k \le h+1$. By the induction hypothesis, the norms of the right-hand sides of (4.26), (4.27) in $H_\gamma^{(2h-2p)m,0}(Q)$, $H_\gamma^{(2h-2p+2)m-m_j-\frac{1}{2},0}(S)$, respectively, are estimated by the right-hand side of (4.26), then so is $\|u_{t^p}\|_{H_\gamma^{(2h-2p+2)m,0}(Q)}$. Hence the estimate (4.5) is valid and the proof is completed.

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