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MECHANICAL SYSTEMS WITH RANDOM PERTURBATIONS ON NON-LINEAR CONFIGURATION SPACES

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The mechanical systems given on non-linear configuration spaces - smooth manifolds - in terms of Newton's second law and subjected to random perturbations of either forces or velocities, are considered. The machinery of mean derivatives is applied for obtaining well-posed description of the systems and for their investigation.

KEY WORDS: mechanical systems; random perturbation of force; random perturbation of velocity; set-valued force; mean derivatives; differential inclusion; Langevin equation.

1. INTRODUCTION

The main aim of this paper is investigation of mechanical systems on non-linear configuration spaces, subjected to the influence of random factors. The paper contains a survey of results obtained in [1] – [7] and some new developments by the authors in this topic. The characteristic feature of our exposition is the use of machinery of mean derivatives according to the ideology of equations and inclusions with mean derivatives suggested in [8] – [10]. Preliminary results and notion can be found in [11] – [14].

It is a well-known fact that a second order differential equation $\ddot{x}(t) = \bar{\alpha}(t, x(t), \dot{x}(t))$ expressing the Newton's law in $\mathbb{R}^n$, is represented as a first order system on the space of doubled dimension

$$\begin{cases} \dot{x}(t) = v(t) \\ \dot{v}(t) = \bar{\alpha}(t, x(t), v(t)). \end{cases} \tag{1.1}$$

We call the first equation of above system horizontal and the second one vertical.

Analogous split takes place in the general case of a mechanical system on non-linear configuration space (smooth manifold) M. For such systems the Newton's law is formulated in terms of covariant derivatives in the form

$$\frac{D}{dt}\dot{m}(t) = \bar{\alpha}(t, m(t), \dot{m}(t)), \tag{1.2}$$

where $\frac{D}{dt}$ is the covariant derivative of Levi—Civitá connection of Riemannian metric on M that determines the kinetic energy of system. Here Newton's law (1.2) is equivalent to equation

$$\frac{d}{dt}(m(t), \dot{m}(t)) = Z(m(t), \dot{m}(t)) + \bar{\alpha}^l(t, m(t), \dot{m}(t)) \tag{1.3}$$

with special vector field (second order differential equation) in the right-hand side on the phase space (tangent bundle) TM where Z is the Levi-Civitá geodesic spray that is horizontal (belongs to the connection), and $\bar{\alpha}^l(t, m(t), \dot{m}(t))$ is the vertical lift of vector force field $\bar{\alpha}(t, m(t), \dot{m}(t))$ that is vertical (belongs to the vertical subspace).

Note that a random perturbation in Newton's law can arise in the horizontal component, in the vertical component and in the both ones. The vertical perturbation means the perturbation of force field while the horizontal one means the perturbation of velocity. All the cases are physically reasonable but they require essentially different methods for their investigation. It should be also pointed out that under random perturbations, the Newton's law becomes a random differential equation. Here we present such equations in terms of mean derivatives (see below). Taking into account various possibilities for constructing second order mean derivatives (forward, backward, mixed, etc.), this yields equations of motion from different parts of physics.

In this paper we deal with the Newton's law in terms of forward mean derivatives. Its physical meaning is the description of motion of ordinary mechanical systems with random perturbations. We consider both the perturbations of forces and of velocities.

The notion of mean derivatives was introduced by Edward Nelson (see [15, 16, 17]) for the needs of stochastic mechanics (a version of quantum mechanics). The equation of motion in this theory (called the Newton—Nelson equation) was the first example of equations in mean derivatives. Later it turned out that the equations in mean derivatives arose also in the description of motion of viscous incompressible fluid (see, e.g., [11, 12, 13, 14]), in the description of Navier—Stokes vortices [18], etc.

In all above-mentioned cases the solutions of the equations were supposed to be Itô diffusion type processes (or even Markov diffusion processes) whose diffusion summand was given a priory since the classical Nelson's mean derivatives yield, roughly speaking, only the drift term (forward, backward, etc.) of a stochastic process. In [8, 9], on the basis a slight modification of some Nelson's idea, a new type of mean derivative is introduced that is responsible for diffusion term. Then it becomes possible in principle to recover a stochastic process from its mean derivatives.

In Section 2 we describe the main construction of mean derivatives.

In Sections 3 and 4 we deal with the so-called Langevin equations and inclusions on manifolds.

The Langevin's equation describes mechanical systems with both deterministic and random forces which have comparable magnitudes (i.e., neither the deterministic nor random part can be neglected) where the random force is a transformed white noise. Examples of such processes are well known in physics (say, the physical Brownian motion is a process of such sort). One can easily realize that in this case the trajectories of the process are a.s. C^1-smooth. This makes the analysis of such systems technically much simpler than that of general ones.

In Section 3, we introduce Langevin's equation on a Riemannian manifold and reduce it to the velocity hodograph equation, which is an equation in a single tangent (i.e., vector) space. This enables us to apply some standard results to carry out a detailed analysis. We study also an important particular case of Langevin's equation: the equation describing the so-called Ornstein-Uhlenbeck processes arising, for example, in the mathematical model of physical Brownian motion [16, 19, 20]. Sometimes, only the latter is called the Langevin equation, whereas that applicable in a more general context is said to be the generalized Langevin equation.

In Section 4 we study the case where the force field in Langevin equation is set-valued (i.e., it is constructed from essentially discontinuous force or the force with feedback control) and so the equation turns into differential inclusion that is well posed in terms of mean derivatives.

Throughout Sections 3 and 4, all Riemannian manifolds are assumed to be complete, not necessarily uniformly or stochastically complete.

In Section 5 we investigate mechanical systems with random perturbations of velocities motivated by motion of a particle, subjected to a deterministic force, that in addition moves with an enveloping media with random influence. First we consider the systems in R^n with single-valued and set-valued forces. The systems on manifolds are investigated under some more restrictive assumptions. In particular, we suppose the manifold to be stochastically complete.

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2. MEAN DERIVATIVES

Consider a stochastic process $\xi(t)$, $t \in [0,T]$, given on a certain probability space $(\Omega, \mathcal{F}, \mathrm{P})$, taking values in $\mathbb{R}^n$ and such that $\xi(t)$ is an L_1 random element for all t.

It is known that such a process determines 3 families of σ-subalgebras of the σ-algebra $\mathcal{F}$:

(i) "the past" $\mathcal{P}_t^\xi$ generated by preimages of Borel sets from $\mathbb{R}^n$ under all mappings $\xi(s):\Omega \to \mathbb{R}^n$ for $0 \le s \le t$;

(ii) "the future" $\mathcal{F}_t^\xi$ generated by preimages of Borel sets from $\mathbb{R}^n$ under all mappings $\xi(s):\Omega \to \mathbb{R}^n$ for $t \le s \le T$;

(iii) "the present" ("now") $\mathcal{N}_t^\xi$ generated by preimages of Borel sets from $\mathbb{R}^n$ under the mapping $\xi(t):\Omega \to \mathbb{R}^n$.

All the above families we suppose to be complete, i.e., containing all sets of probability zero.

For the sake of convenience we denote by E_t^ξ the conditional expectation $E(\cdot \mid \mathcal{N}_t^\xi)$ with respect to the "present" $\mathcal{N}_t^\xi$ for $\xi(t)$.

Note that, generally speaking, a.s. the sample trajectories of $\xi(\cdot)$ are not differentiable and so we cannot determine the derivative of $\xi(\cdot)$ in the ordinary way. According to Nelson (see e. g. [15, 16, 17]) we give the following:

Definition 2.1. *The forward mean derivative $D\xi(t)$ of the process $\xi(t)$ at the moment t is an L^1-random variable of the form*

$$D\xi(t) = \lim_{\Delta t \to +0} E_t^\xi \left(\frac{\xi(t+\Delta t) - \xi(t)}{\Delta t} \right) \quad (2.1)$$

where the limit is assumed to exist in $L^1(\Omega, \mathcal{F}, \mathrm{P})$ and $\Delta t \to +0$ means that $\Delta t \to 0$ and $\Delta t > 0$.

From the properties of the conditional expectation it follows that $D\xi(t)$ is expressed as the composition of $\xi(t)$ and the Borel measurable vector field, namely the regression

$$Y^0(t,x) = \lim_{\Delta t\to+0} E\left(\frac{\xi(t+\Delta t)-\xi(t)}{\Delta t} \mid \xi(t)=x\right) \quad (2.2)$$

on $\mathbb{R}^n$, i.e., $D\xi(t) = Y^0(t,\xi(t))$.

The mean derivative of Definition 2.1 is a particular case of the notion determined as follows. Let $x(t)$ and $y(t)$ be L^1-stochastic processes in F defined on $(\Omega,\mathcal{F},\mathrm{P})$. Introduce y-forward derivative of $x(t)$ by the formula

$$D^y x(t) = \lim_{\Delta t\to+0} E_t^y\left(\frac{x(t+\Delta t)-x(t)}{\Delta t}\right). \quad (2.3)$$

Assume $\xi(t)$ to be an Ito process of diffusion type (see, e.g., [21]) of the form

$$\xi(t) = \xi_0 + \int_0^t \beta(s)ds + \int_0^t A(s)dw(s) \quad (2.4)$$

It should be noticed that $\xi(t)$ can be neither a diffusion nor a Markov process.

Lemma 2.2. *For $\xi(t)$ of type (2.4) $D\xi(t)$ exists and is equal to $E_t^\xi(\beta(t))$.*

Proof. Evidently $D(\xi_0 + \int_0^t\beta(s)ds + \int_0^t A(s)dw(s)) = D^\xi(\int_0^t\beta(s)ds) + D^\xi(\int_0^t A(s)dw(s))$. Since $\int_0^t A(s)dw(s)$ is a martingale with respect to $\mathcal{P}_t^\xi$, $D^\xi(\int_0^t A(s)dw(s)) = 0$. Then $D^\xi(\int_0^t \beta(s)ds) = E_t^\xi(\beta(t))$. ■

Thus by Lemma 2.2 the forward mean derivative gives information about the drift of an Itô process. Following [8, 9] we introduce a new mean derivative D_2, called quadratic, that is responsible for diffusion term of a process. It is a slight modification of a certain Nelson's idea from [17].

Definition 2.3. *For an L^1-stochastic process ξ $t\in[0,T]$, its quadratic mean derivative $D_2\xi(t)$ is defined by the formula*

$$D_2\xi(t) = \lim_{\Delta t\to+0} E_t^\xi\left(\frac{(\xi(t+\Delta t)-\xi(t))\otimes(\xi(t+\Delta t)-\xi(t))}{\Delta t}\right), \quad (2.5)$$

where $\otimes$ denotes the tensor product and the limit is supposed to exist in $L_1(\Omega,\mathcal{F},\mathrm{P})$.

Denote by $S_+(n)$ the set of symmetric positive definite $n\times n$ matrices and by $\bar{S}_+(n)$ the set of symmetric positive semi-definite matrices (the closure of $S_+(n)$ in the space of all symmetric matrices $S(n)$).

We emphasize that the tensor product as in (2.5) is a symmetric positive semi-definite matrix so that $D_2\xi(t)$ is a function with values in $\bar{S}_+(n)$.

From the properties of conditional expectation it follows that there exist a Borel mapping $\alpha(t,x)$ from $[0,T]\times\mathbb{R}^n$ to $\bar{S}_+(n)$ such that $D_2\xi(t) = \alpha(t,\xi(t))$. As well as above we call $\alpha(t,x)$ the regression.

Theorem 2.4. *Let $\xi(t)$ be a diffusion type process. Then $D_2\xi(t) = E_t^\xi[\alpha(t)]$ where $\alpha(t) = A(t)A^*(t)$, $A^*(t)$ is the transposed matrix $A(t)$ and $A(t)A^*(t)$ is the matrix product. In particular, if $\xi(t)$ is a diffusion process, $D_2\xi(t) = \alpha(t,\xi(t))$ where α is the diffusion coefficient.*

Proof. From direct calculation it follows that the components of $(\xi(t+\Delta t)-\xi(t))\otimes(\xi(t+\Delta t)-\xi(t))$ are elements of the matrix $(\xi(t+\Delta t)-\xi(t))(\xi(t+\Delta t)-\xi(t))^*$ where we use the matrix multiplication of the column-vector $(\xi(t+\Delta t)-\xi(t))$ and the row-vector $(\xi(t+\Delta t)-\xi(t))^*$ (i.e., transposed $(\xi(t+\Delta t)-\xi(t))$). The product is a symmetric semi-positive definite matrix. Since $\xi(t+\Delta t)-\xi(t) = \int_t^{t+\Delta t} a(s)ds + \int_t^{t+\Delta t} A(s)dw(s)$, taking into account the properties of Lebesgue and Itô integrals one can see that $(\xi(t+\Delta t)-\xi(t))(\xi(t+\Delta t)-\xi(t))^*$ is approximated by $a(t)a(t)^*(\Delta t)^2 + (a(t)\Delta t)(A(t)\Delta w(t))^* + (A(t)\Delta w(t))(a(t)\Delta t)^* + A(t)A(t)^*\Delta t$. Thus we see that only $A(t)A(t)^*\Delta t$ is infinitesimal of the same order as Δt while the other summands are infinitesimals of order higher than Δt. Applying formula (2.5) obtain the assertion of Theorem since $AA^* = \alpha$ (see above). ■

Let Z(t,x) be a C^2-smooth vector field on $\mathbb{R}^n$, and $\xi(t)$ be a stochastic process in $\mathbb{R}^n$.

Definition 2.5. *The forward $DZ(t,\xi(t))$ mean derivative of Z along $\xi(\cdot)$ at t is the L^1-limits of the form*

$$DZ(t,\xi(t)) = \lim_{\Delta t\to+0} E_t^\xi\left(\frac{Z(t+\Delta t,\xi(t+\Delta t))-Z(t,\xi(t))}{\Delta t}\right) \quad (2.6)$$

Of course $DZ(t,\xi(t))$ can be presented as compositions of $\xi(t)$ with a certain Borel vector fields on $\mathbb{R}^n$. This vector field (regression) will be also denoted by DZ.

3. LANGEVIN EQUATION AND ORNSTEIN—UHLENBECK PROCESSES ON MANIFOLDS

Everywhere in this section we deal with porcesses given on a certain finite time interval $[0,l]\subset\mathbb{R}$.

Let M be a manifold and TM be its tangent bundle.

Definition 3.1. *A map* $\bar{\alpha} : R \times TM \to TM$ *(either single or set-valued), such that*

$$\pi\bar{\alpha}(t,m,X) = \pi(m,X) = m$$

(π *is the natural projection of* TM *onto* M*) is called vector force field.*

Consider a mechanical system on non-linear configuration space (see e.g., [13, 14]), i.e., a Riemannian manifold M and a vector force field on it. Newton's law (1.2) is the equation of motion fot such system. The Riemannian metric on M is assumed to be complete, i.e., a free particle on M does not go to infinity in finite time. In addition to Definition 3.1 of vector force field we give the following

Definition 3.2. *A map* a *from* $R \times TM$ *to the bundle of* $(1,1)$*-tensors over* M *(either single or set-valued), such that* $\pi_1 a(t,m,X) = \pi(m,X) = m$ *(*π_1 *is the projection of the bundle of* $(1,1)$*-tensors onto* M*) will be called tensor force field.*

Recall that a $(1,1)$-tensor at $m \in M$ is a linear operator in $T_m M$. Thus it is evident that for a tensor force field $\bar{\alpha}(t,m,X)$ and a vector field $Y(m)$ on M the composition $a(t,m,X) \circ Y(m)$ is a vector force field.

Let $\bar{\alpha}(t,m,X)$ be a vector force field and $A(t,m,X)$ a $(1,1)$-tensor field on M. In other words, for every $t \in [0,l]$, $m \in M$, and $X \in T_m M$, we have a vector $\bar{\alpha}(t,m,X) \in T_m M$ and a linear operator $A(t,m,X) T_m M \to T_m M$. Specify a Wiener process w on the tangent spaces to M and denote by $\dot{w}$ the white noise of w. Then the Langevin equation describes the evolution of a system with the force field:

$$\bar{\alpha}(t,m,X) + A(t,m,X)\dot{w}. \tag{3.1}$$

More formally, the equation of motion must read

$$\frac{D}{dt}\dot{\xi}(t) = \bar{\alpha}(t,\xi(t),\dot{\xi}(t)) + A(t,\xi(t),\dot{\xi}(t))\dot{w}(t) \tag{3.2}$$

but this expression makes sense only by means of distributions.

Our first goal is to give a rigorous meaning to (3.2) without using distributions. We do it in terms of forward mean derivatives the construction of which can be slightly simplified in the case under consideration.

We assume that $\bar{\alpha}(t,m,X)$ and $A(t,m,X)$ are continuous jointly in all variables and that these fields have linear growth in X. In other words, there exists a constant $K > 0$ such that

$$\|\bar{\alpha}(t,m,X)\| + \|A(t,m,X)\| < K(1 + \|X\|) \tag{3.3}$$

for all $t \in [0,l]$, $m \in M$, and $X \in T_m M$, where the norm is given by the Riemannian metric.

From physical reasons one can derive that a process subjected to force (3.1), a.s. has continuous velocities and as a consequence it a.s. has C^1-smooth sample paths. Below we shall show that, indeed, solutions of Langevin equation exist in the class of processes with C^1-smooth sample paths. That is why we start with some features of such processes.

Let $\xi(t)$ be a stochastic process on M with a.s. C^1-smooth sample paths, given on a certain probability space $(\Omega, \mathcal{F}, \mathrm{P})$, and let a vector field Y be given on M. As well as above, by $\Gamma_{t,s}$ we denote the operator of parallel translation along a C^1-smooth curve $x(\cdot)$ from $x(s)$ to $x(t)$.

Definition 3.3. *Covariant forward mean derivative of vector field* Y *along the process* $\xi(t)$ *on* M *with a.s.* C^1*-smooth sample paths at time instant* t *is the* L^1 *random element of the form*

$$\mathbf{D}Y(t,\xi(t)) = $$
$$= \lim_{\Delta t \downarrow 0} E_t^{\xi}\left(\frac{\Gamma_{t,t+\Delta t}Y(t+\Delta t, \xi(t+\Delta t)) - Y(t,\xi(t))}{\Delta t}\right). \tag{3.4}$$

where $\Gamma_{t,s}$ *is the ordinary parallel translation along* C^1*-smooth curves.*

Let $I = [0,l]$ be an interval in $\mathbb{R}$ and $v : I \to T_{m_0}M$ be a continuous curve.

Theorem 3.4 *(see [11]—[14]). There exists a unique* C^1*-curve* $\gamma : I \to M$ *such that* $\gamma(0) = m_0$ *and the tangent vector* $\dot{\gamma}(t)$ *is parallel to the vector* $v(t) \in T_{m_0}M$ *for every* $t \in I$.

In what follows, we denote by $\mathcal{S}v(\cdot)$ the curve γ constructed as above beginning with v.

Consider a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and a non-decreasing family of complete σ-subalgebras $\mathcal{B}_t$ of $\mathcal{F}$. In a certain tangent space $T_{m_0}M$ introduce a Wiener process $w(t)$ adapted to $\mathcal{B}_t$, and an Itô diffusion type process $v(t)$ of the form $v(t) = \int_0^t b(s)ds + \int_0^t B(s)dw(s)$ with $b(t)$ and $B(t)$ a.s. having continuous sample paths. In particular this means that $v(t)$ is non-anticipative with respect to $\mathcal{B}_t$ and a.s. has continuous sample paths. Thus we can apply operator $\mathcal{S}$ to sample paths of $v(t)$. Then we obtain the process $\xi(t) = Sv(t)$ having C^1-smooth sample paths. Recall that $SC^0([0,l], T_{m_0}M) \to C^1_{m_0}([0,l], M)$ is continuous.

Lemma 3.5. *The process* $\mathcal{S}v(t)$ *is nonanticipative with respect to* $\mathcal{B}_t$.

Consider a special case of the probability space: $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathrm{P}})$ where $\bar{\Omega} = C^0([0,l], T_{m_0}M)$, $\bar{\mathcal{F}}$ is the σ-algebra generated by cylinder sets and the measure $\bar{\mathrm{P}}$ will be the one generated by a certain stochastic process in $T_{m_0}M$. In this case we shall deal with the family $\bar{\mathcal{B}}_t$ of σ-sub-algebras of $\bar{\mathcal{F}}$ where for a specified t the σ-sub-algebra $\bar{\mathcal{B}}_t$ is generated by cylinder sets with bases on $[0,t]$.

Lemma 3.6. *The process $\mathcal{S}v(t)$ is nonanticipative with respect to $\bar{\mathcal{B}}_t$.*

Proof. Indeed, if the curves $v_1(\cdot)$ and $v_2(\cdot)$ from $\bar{\Omega} = C^0([0,l], T_{m_0}M)$ coincide at all $t \in [0, l_0]$ where $0 < l_0, < l$, then $\mathcal{S}v_1(t)$ coincides with $\mathcal{S}v_2(t)$ for $t \in [0, l_0]$ by construction of operator $\mathcal{S}$. From this we obtain the assertion of Lemma 3.6. ■

Consider the vector field $\dot{\xi}(t)$ along $\xi(t) = \mathcal{S}v(t)$.

Theorem 3.7.

$$\mathbf{D}\dot{\xi}(t) = E_t^{\xi}(\Gamma_{t,0} b(t)).$$

Proof. From the properties of parallel translation and from construction of $\xi(t)$ it follows that

$$E_t^{\xi}(\Gamma_{t,t+\triangle t}\dot{\xi}(t+\triangle t) - \dot{\xi}(t)) =$$
$$= E_t^{\xi}\left(\Gamma_{t,0}\left(\int_t^{t+\triangle t} b(s)ds + \int_t^{t+\triangle t} B(s)dw(s)\right)\right).$$

Note that $\mathcal{N}_t^{\xi}$ is a σ-subalgebra in $\mathcal{P}_t^{v}$. Since the Itô integral $\int_t^{t+\triangle t} B(s)dw(s)$ is a martingale with respect to $\mathcal{P}_t^{v}$, by the properties of conditional expectation we obtain that

$$E_t^{\xi}\left(\int_t^{t+\triangle t} B(s)dw(s)\right) = 0.$$

The Theorem follows. ■

Along a process $\xi(t) = Sv(t)$ as above we can define the covariant quadratic mean derivative of $\dot{\xi}(t)$ as follows. Introduce the notation $\triangle\dot{\xi}(t) = \Gamma_{t,t+\triangle t}\dot{\xi}(t+\triangle t) - \dot{\xi}(t)$ where (as well as above in this section) $\Gamma_{t,s}$ is the ordinary parallel translation along C^1-smooth curves.

Definition 3.8. *Quadratic mean derivative of $\dot{\xi}(t)$ along $\xi(t) = \mathcal{S}v(t)$ on M at time instant t is a L^1 random element of the form*

$$\mathbf{D}_2\dot{\xi}(t) = \lim_{\triangle t \downarrow 0} E_t^{\xi}\left(\frac{\triangle\dot{\xi}(t) \otimes \triangle\dot{\xi}(t)}{\triangle t}\right),$$

where $\otimes$ is the tensor product and $\Gamma_{t,s}$ is the ordinary parallel translation along C^1-smooth curves.

Theorem 3.9. *Consider a process $\xi(t) = \mathcal{S}v(t)$ with $v(t) = \int_0^t b(s)ds + \int_0^t B(s)dw(s)$ in $T_{m_0}M$ as above. Then $\mathbf{D}_2\dot{\xi}(t) = E_t^{\xi}(\Gamma_{t,0}(B(t)B^*(t)))$ where B^* is the adjoint operator.*

Proof. As well as in the proof of Theorem 3.7, from the properties of parallel translation and from construction of $\xi(t)$ it follows that

$$E_t^{\xi}(\triangle\dot{\xi}(t) \otimes \triangle\dot{\xi}(t)) = E_t^{\xi}(\Gamma_{0,t}(\triangle v(t) \otimes \triangle v(t))),$$

where $\triangle v(t) = \int_t^{t+\triangle t} b(s)ds + \int_t^{t+\triangle t} B(s)dw(s)$. In addition from the properties of Itô integral we obtain that in the expression $\triangle v(t) \otimes \triangle v(t)$ only the summand $(\int_t^{t+\triangle t} B(s)dw(s)) \otimes (\int_t^{t+\triangle t} B(s)dw(s))$ is infinitesimal of the same order as $\triangle t$ while all other summand are infinitesimals of higher order than $\triangle t$. Now the Theorem follows from Definition 3.8 and from the properties of Itô integral. ■

Definition 3.10. *Langevin equation with force field (3.1) is the system*

$$\begin{cases} \mathbf{D}\dot{\xi}(t) = \bar{\alpha}(t, \xi(t), \dot{\xi}(t)) \\ \mathbf{D}_2\dot{\xi}(t) = A(t, \xi(t), \dot{\xi}(t))\, A^*(t, \xi(t), \dot{\xi}(t)). \end{cases} \tag{3.5}$$

Let $\xi(t)$ be a stochastic process with values in M which is nonanticipative with respect to $\mathcal{B}_t$ and such that the sample trajectories of ξ are a.s. C^1-smooth and $\xi(0) = m_0 \in M$. For the sake of convenience denote $\Gamma_{0,t}$ by Γ and so

$$\Gamma\bar{\alpha}(t, \xi(t), \dot{\xi}(t)) \text{ and } \Gamma A(t, \xi(t), \dot{\xi}(t))$$

are obtained by the parallel translation of

$$\bar{\alpha}(t, \xi(t), \dot{\xi}(t)) \text{ and } A(t, \xi(t), \dot{\xi}(t)),$$

respectively, along $\xi(\cdot)$ from the point $\xi(t)$ to $\xi(0) = m_0$, where $\bar{\alpha}$ and A are the coefficients of force field (3.1). The processes $\Gamma\bar{\alpha}(t, \xi, \dot{\xi})$ and $\Gamma A(t, \xi, \dot{\xi})$ take values in $T_{m_0}M$ and $L(T_{m_0}M, T_{m_0}M)$, respectively, and their trajectories are a.s. continuous, for so are the fields $\bar{\alpha}(t, m, X)$ and $A(t, m, X)$. Since parallel translation preserves the Riemannian norm, it follows from (3.3) that

$$\left\|\Gamma\bar{\alpha}(t, \xi(t), \dot{\xi}(t))\right\| + \left\|\Gamma A(t, \xi(t), \dot{\xi}(t))\right\| <$$
$$< K\left(1 + \left\|\Gamma\dot{\xi}(t)\right\|\right). \tag{3.6}$$

Lemma 3.11. *The processes $\Gamma\bar{\alpha}(t, \xi(t), \dot{\xi}(t))$ and $\Gamma A(t, \xi(t), \dot{\xi}(t))$ are nonanticipative with respect to $\mathcal{B}_t$.*

The lemma is a consequence of the fact that the parallel translation operator Γ is continuous on the space of C^1-curves equipped with the C^1-topology and of our assumptions that ξ is nonanticipative and the fields $\bar{\alpha}$ and A are both continuous.

By Lemma 3.11, we can define the process $z(t)$ in $T_{m_0}M$ as

$$z(t) = \int_0^t \Gamma\bar{\alpha}(\tau, \xi(\tau), \dot{\xi}(\tau))d\tau + \\ +\int_0^t \Gamma A(\tau, \xi(\tau), \dot{\xi}(\tau))dw(\tau), \qquad (3.7)$$

where the second term on the right-hand side is the Itô integral. It is clear that $z(t)$ given by (3.7) is nonanticipative with respect to $\mathcal{B}_t$ and almost surely has continuous trajectories.

Setting $v(t) = z(t)$, we obtain Langevin's equation (3.5) in the integral form:

$$\xi(t) = S(\int_0^t \Gamma\bar{\alpha}(\tau, \xi(\tau), \dot{\xi}(\tau))d\tau + \\ +\int_0^t \Gamma A(\tau, \xi(\tau), \dot{\xi}(\tau))dw(\tau) + \bar{C}). \qquad (3.8)$$

Indeed, one can easily see that a process satisfying (3.8), satisfies also (3.5).

Definition 3.12. *We say that (3.8) has a weak solution on $[0,l] \subset R$ with initial conditions $\xi(0) = m_0$, $\dot{\xi}(0) = C$ if there exist a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, a stochastic process $\xi(t)$ with a.s. C^1-smooth sample paths, defined on $(\Omega, \mathcal{F}, \mathrm{P})$ and valued in M with initial condition $\xi(0) = m_0$ and $\dot{\xi}(0) = C$, a Wiener process $w(t)$ in $\mathbb{R}^n$, defined on $(\Omega, \mathcal{F}, \mathrm{P})$ and adapted to $\xi(t)$, such that for all $t \in [0,l]$ P-a.s. (3.8) is fulfilled.*

Definition 3.13. *We say that (3.8) has a strong solution on $[0,l] \subset R$ with initial conditions $\xi(0) = m_0$, $\dot{\xi}(0) = C$ if on every probability space $(\Omega, \mathcal{F}, \mathrm{P})$ such that it admits a Wiener process, and for any Wiener process $w(t)$ in $\mathbb{R}^n$, defined on $(\Omega, \mathcal{F}, \mathrm{P})$, there exists a stochastic process $\xi(t)$, non-anticipative with respect to $w(t)$ and having a.s. C^1-smooth sample paths, that is defined on $(\Omega, \mathcal{F}, \mathrm{P})$ and valued in M with initial condition $\xi(0) = m_0$, such that for all $t \in [0,l]$ P-a.s. (3.8) is fulfilled.*

Let $m(t)$, $t \in I$, be a trajectory of the mechanical system, i.e., a solution of (1.2).

Definition 3.14. *The velocity hodograph of the trajectory $m(t)$ is the curve $vI \to T_{m(0)}M$ such that $v(t)$ is parallel to $\dot{m}(t)$ along $m(\cdot)$.*

The equation of the velocity hodograph corresponding to (3.8) is

$$v(t) = \int_0^t \Gamma\bar{\alpha}\left(t, \mathcal{S}v(t), \frac{d}{dt}\mathcal{S}v(t)\right)d\tau + \\ + \int_0^t \Gamma A\left(t, \mathcal{S}v(t), \frac{d}{dt}\mathcal{S}v(t)\right)dw(\tau) + \bar{C}. \qquad (3.9)$$

It is clear that the vector $\Gamma\bar{\alpha}\left(t, \mathcal{S}x(t), \frac{d}{dt}\mathcal{S}x(t)\right)$ and the tensor $\Gamma A\left(t, \mathcal{S}x(t), \frac{d}{dt}\mathcal{S}x(t)\right)$ are well-posed along any curve $x(\cdot) \in C^0(I, T_{m_0}M)$ and continuous on the space $\mathbb{R} \times C^0(I, T_{m_0}M)$. By construction and by the properties of parallel translation, we have

$$\left\|\frac{d}{dt}\mathcal{S}(x(t))\right\| = \|x(t)\|,$$

and, therefore, due to (3.6),

$$\left\|\Gamma\bar{\alpha}\left(t, \mathcal{S}x(t), \frac{d}{dt}\mathcal{S}x(t)\right)\right\| + \\ +\left\|\Gamma A\left(t, \mathcal{S}x(t), \frac{d}{dt}\mathcal{S}x(t)\right)\right\| \le K(1 + \|x(t)\|). \qquad (3.10)$$

Lemma 3.15. *$\Gamma\bar{\alpha}\left(t, \mathcal{S}x(t), \frac{d}{dt}\mathcal{S}x(t)\right)$ and $\Gamma A\left(t, \mathcal{S}x(t), \frac{d}{dt}\mathcal{S}x(t)\right)$ are non-anticipative with respect to the family $\bar{B}_t$ from Lemma 3.6.*

The assertion of Lemma 3.15 follows from constructions of $\Gamma\bar{\alpha}\left(t, \mathcal{S}x(t), \frac{d}{dt}\mathcal{S}x(t)\right)$ and of $\Gamma A\left(t, \mathcal{S}x(t), \frac{d}{dt}\mathcal{S}x(t)\right)$ and from the properties of parallel translation as well as from Lemma 3.6.

Equation (3.9) is an Itô stochastic differential equation of diffusion type on the linear space $T_{m_0}M$. Thus we needn't introduce special notions of strong and weak solutions of (3.9).

It is clear that $v(t)$ and the Wiener process $w(t)$ in $T_{m_0}M$ satisfy (3.9) if and only if $\mathcal{S}v(t)$ (taking values in M) and $w(t)$ satisfy (3.8). Observe also that $Sv(t)$ is defined on the same probability space and has the same measurability properties with respect to $w(t)$ as $v(t)$. Thus, we have proved

Theorem 3.16. *The process $v(t)$ is a strong (respectively, weak) solution of (3.9) if and only if $Sv(t)$ is a strong (respectively, weak) solution of (3.8).*

Remark 3.17. *Let us specify a realization of $w(t)$ in $T_{m_0}M$. Applying to it the parallel translation along $\mathcal{S}v(\cdot)$, we obtain realizations of $w(t)$ in all spaces $T_{\mathcal{S}v(\cdot)}M$. These realizations give rise to a force field defined along the trajectory.*

Theorem 3.18. *Assume that $\bar{\alpha}(t,m,X)$ and $A(t,m,X)$ are jointly continuous in all variables and satisfy (3.3). Then on $[0,l]$, there exists a weak solution of equation (3.8) for any initial conditions $\xi(0) = m_0$ and $\dot{\xi}(0) = \bar{C} \in T_{m_0}M$.*

Proof. First, we pass to (3.9), which is equivalent to (3.8). Note that (3.9) is a diffusion type equation on a vector space. Recall that this means that the coefficients of (3.9) depend on the past, i.e., on the entire trajectory on the interval $[0,t]$. As has been shown, $\Gamma\bar{\alpha}\left(t, \mathcal{S}x(t), \frac{d}{dt}\mathcal{S}x(t)\right)$ and $\Gamma A\left(t, \mathcal{S}x(t), \frac{d}{dt}\mathcal{S}x(t)\right)$ are well defined and continuous on $\mathbb{R}\times C^0([0,l],T_{m_0}M)$. Moreover, they satisfy (3.10), the linear growth condition and by Lemma 3.15 they are not anticipative with respect to the family of σ-sub-algebras $\bar{\mathcal{B}}_t$. Thus, the standard existence theorem in linear spaces (see, e.g., Chapter 3, Section 2 of [21] or Section 19.3.8 of [22]) guarantees that a weak solution of (3.9) exists. To complete the proof it suffices to apply Theorem 3.16. ■

The following results can also be proved by passing to (3.9) and applying the results of the standard theory of stochastic equations on vector spaces [21, 22].

Theorem 3.19. *Let $\bar{\alpha}(t,m,X)$ and $A(t,m,X)$ be as in Theorem 3.18 Assume that the operator $A(t,m,X)$ is invertible for all t, m, and X. If a solution of the equation*

$$\xi(t)=\mathcal{S}\left(\int_0^t \Gamma A(\tau,\xi(\tau),\dot{\xi}(\tau))dw(\tau)\right) \quad (3.11)$$

is weakly unique, then so is a solution of (3.8).

Theorem 3.20. *Let $\bar{\alpha}(t,m,X)$ be continuous jointly in all variables, satisfy (3.3), and such that the solution of the Cauchy problem for (1.2) is unique. Also, let $A_\varepsilon(t,m,X)$, where $\varepsilon\in(0,\delta)$ and $\delta>0$, be jointly continuous in ε, t, m, and X and satisfy (3.3) with K independent of ε. Assume, in addition, that*

(i) $A_0=0$;

(ii) $\lim\limits_{\varepsilon\to0}A_\varepsilon\to0$ *uniformly on every compact in* $[0,l]\times TM$;

(iii) *a solution of the equation*

$$\xi(t)=\mathcal{S}\Bigg(\int_0^t \Gamma\bar{\alpha}(\tau,\xi(\tau),\dot{\xi}(\tau))d\tau+ \\ +\int_0^t \Gamma A_\varepsilon(\tau,\xi(\tau),\dot{\xi}(\tau))dw(\tau)+\bar{C}_\varepsilon\Bigg) \quad (3.12)$$

is weakly unique for some $\bar{C}_\varepsilon$ such that $\lim\limits_{\varepsilon\to0}\bar{C}_\varepsilon=\bar{C}$. Then the measures on $C^1_{m_0}([0,l],M)$ corresponding to the solutions of (3.12) weakly converge as $\varepsilon\to0$ to the measure concentrated on the unique solution of (1.2).

Example 3.21. *Let $A=\varepsilon I$, where I is the identity operator. Then it is clear that a solution of (3.11) is unique. Thus, for $\bar{\alpha}$ as before, the equation*

$$\xi(t)=\mathcal{S}\left(\int_0^t \Gamma\bar{\alpha}(\tau,\xi(\tau),\dot{\xi}(\tau))d\tau+\varepsilon w(t)+C\right)$$

has a unique solution. If, for example, $\bar{\alpha}$ is locally Lipschitz in m and X, then Theorem 3.20 holds true for the latter equation.

It is known that equation (3.8) has a strongly unique strong solution, provided that the coefficients of diffusion type equation (3.8) satisfy a Lipschitz type condition (see e.g., [21, 22]). However, the existence of a strong solution is rather hard to prove in the general case where the coefficients involve the operators Γ and $\mathcal{S}$. The reason is that Γ and $\mathcal{S}$ are defined by means of parallel translation and, as a consequence, we have a condition imposed on the entire mechanical system, rather than just on the force field.

On the other hand, the existence can easily be verified for certain particular force fields. Here we consider three examples of such fields:

(i) The drag force:

$$\bar{\alpha}(t,m,X)=\phi(t,\|X\|)\cdot\hat{a}_m(X),$$
$$A(t,m,X)=\Psi(t,\|X\|)\cdot\hat{A}_m(X),$$

where ϕ and Ψ are scalar functions, $\hat{a}$ is a $(1,1)$-tensor field with $\nabla\hat{a}=0$, and $\hat{A}$ is a field of operators $\hat{A}_m T_m M\to L(T_mM)$ parallel along every curve in M. (Note that the equation $\nabla\hat{a}=0$ means, in fact, a restriction of the same kind as that imposed on $\hat{A}$: the operators $\hat{a}_m T_m M\to T_mM$ are parallel along every curve.) For example, one may take $\hat{a}=\pm I$ or, if M is an oriented two-dimensional manifold, then $\hat{a}_m$ may be the rotation by a fixed angle. The same operators can be taken as examples of $\hat{A}$ if we assume in addition that $\hat{A}_m(X)$ is independent of X (i.e., $\hat{A}_m$, regarded as a function of X, is constant).

(ii) A particular case of (i) involving friction and constant diffusion:

$$\bar{\alpha}(t,m,X)=-b(t)\cdot X,\ A(t,m,X)=\phi(t)\cdot\hat{A}_m,$$

where the friction coefficient $b\geq0$ is a real-valued function of time and $\hat{A}$ is a $(1,1)$-tensor field with $\nabla\hat{A}=0$.

(iii) A force given in a "stationary coordinate system". Let $\bar{\alpha}_{m_0}(t)T_{m_0}M\to T_{m_0}M$ and $A_{m_0}(t)T_{m_0}M\to$ $\to L(T_{m_0}M)$, $t\in[0,l]$ be given. The operators $\bar{\alpha}$ and A at other points of the trajectory $\xi(t)$ are obtained by the parallel translation of $\bar{\alpha}_{m_0}$ and A_{m_0} along $\xi(\cdot)$. (See, e.g., [13, 14] for a mechanical interpretation of parallelism.)

Theorem 3.22. *Let $\bar{\alpha}$ and A be as in (i)-(iii). Assume also that $\bar{\alpha}_{m_0}$ and A_{m_0} are Lipschitz in $X\in T_{m_0}M$ and satisfy (3.3), the linear growth condition. Then (3.8) has a strongly unique strong solution on I.*

Proof. Under the hypotheses of the theorem, equation (3.9) on $T_{m_0}M$ is equivalent to the following one:

$$v(t)=\int_0^t \bar{\alpha}(\tau,m_0,v(\tau))d\tau+$$
$$+\int_0^t A(\tau,m_0,v(\tau))dw(\tau)+\bar{C}. \qquad (3.13)$$

This equation has a strongly unique strong solution defined on $[0,l]$. The initial velocity $\bar{C}$ can be viewed as a random vector measurable with respect to the σ-algebra $\mathcal{B}_0$ [21,22]. To finish the proof it suffices to apply Theorem 3.16. ■

Note that if $\bar{\alpha}$ and A are as in (ii), then the hypotheses of Theorem 3.22 are automatically satisfied, provided that b and ϕ are bounded. In this case, the solution $v(t)$ of (3.13) and the solution $\mathcal{S}v(t)$ of Langevin's equation are called the *velocity Ornstein—Uhlenbeck process* and *the coordinate Ornstein—Uhlenbeck process*, respectively.

Note that the assumption that b and ϕ are bounded can be omitted in the hodograph equation for Ornstein—Uhlenbeck processes so that the velocity process exists on a random interval up to the so-called explosion time.

Recall that Ornstein—Uhlenbeck processes describe the Brownian motion in a medium with a drag force. A detailed discussion of this matter can be found in [16]. Ornstein—Uhlenbeck processes on manifolds are also discussed in [19].

Let $v(t)$ be a solution of (3.13). Denote by $Ev(t)$ the mathematical expectation of $v(t)$ in $T_{m_0}M$.

Definition 3.23. *The curve $\mathcal{S}(Ev(t))$ on M is said to be the mathematical expectation of the process $\mathcal{S}v(t)$. The function $E(Ev(t)-v(t))^2$ is called the dispersion of $\mathcal{S}v(t)$.*

It is easy to see that for a system defined in (i) and, in particular, for (ii) the mathematical expectation of a solution of (3.8) satisfies (1.2).

Passing to the hodograph equation and applying the standard results on equations in a vector space, we obtain the following theorem.

Theorem 3.24. *Under the assumptions of Theorem 3.22, the solutions of*

$$\xi(t)=\mathcal{S}\Big(\int_0^t \Gamma\bar{\alpha}(\tau,\xi(\tau),\dot{\xi}(\tau))d\tau+$$
$$+\varepsilon\int_0^t \Gamma A(\tau,\xi(\tau),\dot{\xi}(\tau))dw(\tau)+\varepsilon\bar{C}\Big) \qquad (3.14)$$

converge as $\varepsilon\to 0$ to the solution of (1.2) in the topology of the space

$$\mathcal{S}(C^0([0,l],L_2(\Omega,T_{m_0}M))).$$

The mathematical expectation of ξ uniformly converges to the solution of (1.2).

Here $L^2(\Omega,T_{m_0}M)$ is the space of the square integrable maps from Ω to $T_{m_0}M$. Note that the convergence means that the dispersion of ξ converges uniformly to zero.

4. SET-VALUED FORCES. LANGEVIN TYPE INCLUSIONS

In this section we investigate second order stochastic differential inclusions on Riemannian manifolds that are set-valued analogues of Langevin equations from Section 3. The set-valued force evidently arises in a system with control or may be obtained from a discontinuous force (for instance, the dry friction is considered or the motion takes place in a complicated medium, etc.). Recall that if the force is discontinuous there are well-known methods of transition to a set-valued force (for stochastic differential equations the pioneering paper was probably [23]). Examples of systems having discontinuous forces with random components of the above-mentioned sort are rather usual in physics, e.g., they describe the motion of the physical Brownian particle in a complicated medium. The use of Riemannian manifolds allows one to cover the mechanics on non-linear configuration spaces.

Definition 4.1. *A set-valued map $f:\mathbb{R}\times TM\multimap TM$ such that for any point $(m,X)\in TM$ (this means that $X\in T_mM$, i.e., X is a tangent vector to M at the point $m\in M$) the relation $\pi f(t,m,X)=\pi(m,X)=m$ holds, is called set-valued vector force field.*

A set-valued map a from $\mathbb{R}\times TM$ to the bundle of $(1,1)$-tensors over M such that $\pi_1 a(t,m,X)=\pi(m,X)=m$ (π_1 is the projection of the bundle of $(1,1)$-tensors onto M) will be called set-valued tensor force field.

Now let $\boldsymbol{\alpha}$ and $\mathbf{A}$ be set-valued vector force field and set valued tensor field, respectively. For a stochastic process $\xi(t)$ with a.s. C^1-smooth sample paths consider the set-valued maps $\Gamma\boldsymbol{\alpha}(\tau,\xi(\tau),\dot{\xi}(\tau))$ and $\Gamma\mathbf{A}(\tau,\xi(\tau),\dot{\xi}(\tau))$ sending $[0,l]$ into $T_{m_0}M$ and into the space of linear operators on $T_{m_0}M$ and denote by $\mathcal{P}\Gamma\boldsymbol{\alpha}(\tau,\xi(\tau),\dot{\xi}(\tau))$ and $\mathcal{P}\Gamma\mathbf{A}(\tau,\xi(\tau),\dot{\xi}(\tau))$ the sets of their Borel measurable selectors.

The Langevin inclusion is a system of the form

$$\begin{cases}\mathbf{D}\dot{\xi}(t)\in\boldsymbol{\alpha}(t,\xi(t),\dot{\xi}(t))\\ \mathbf{D}_2\dot{\xi}(t)\in\mathbf{A}(t,\xi(t),\dot{\xi}(t))\mathbf{A}^*(t,\xi(t),\dot{\xi}(t))\end{cases} \qquad (4.1)$$

where $\mathbf{D}$ and $\mathbf{D}_2$ are defined in Section 3 by means of ordinary parallel translation along C^1-smooth curves and $\mathbf{AA}^* = \{AA^* \mid A \in \mathbf{A}\}$. In integral form (4.1) is expressed as

$$\xi(t) \in \mathcal{S}\left(\int_0^t \mathcal{P}\Gamma\boldsymbol{\alpha}(\tau,\xi(\tau),\dot{\xi}(\tau))d\tau + \right.$$

$$\left. + \int_0^t \mathcal{P}\Gamma\mathbf{A}(\tau,\xi(\tau),\dot{\xi}(\tau))dw(\tau) + C\right). \qquad (4.2)$$

Definition 4.2. *We say that (4.2) has a weak solution on* $[0,l] \subset R$ *with initial conditions* $\xi(0) = m_0$, $\dot{\xi}(0) = C$ *if there exist a probability space* $(\Omega,\mathcal{F},\mathrm{P})$, *a stochastic process* $\xi(t)$ *with a.s.* C^1*-smooth sample paths, defined on* $(\Omega,\mathcal{F},\mathrm{P})$ *and valued in* M *with initial condition* $\xi(0) = m_0$ *and* $\dot{\xi}(0) = C$, *a Wiener process* $w(t)$ *in* $\mathbb{R}^n$, *defined on* $(\Omega,\mathcal{F},\mathrm{P})$ *and adapted to* $\xi(t)$, *a single-valued vector field* $\bar{\alpha}(t,m,X)$ *on* M *and a single-valued* $(1,1)$*-tensor field* $A(t,m,X)$ *such that*

(i) for all t *the random vector* $\bar{\alpha}(t,\xi(t),\dot{\xi}(t))$ *belongs to* $\boldsymbol{\alpha}(t,\xi(t),\dot{\xi}(t))$ P*-almost surely (a.s.);*

(ii) for all t *the random tensor* $A(t,\xi(t),\dot{\xi}(t))$ *belongs to* $\mathbf{A}(t,\xi(t),\dot{\xi}(t))$ P*-a.s.;*

(iii) the integrals $\int_0^t \Gamma\bar{\alpha}(\tau,\xi(\tau),\dot{\xi}(\tau))d\tau$ *and* $\int_0^t \Gamma A(\tau,\xi(\tau),\dot{\xi}(\tau))dw(\tau)$ *are well-posed for* $\xi(t)$, $w(t)$, $\bar{\alpha}$ *and* A

and for all $t \in [0,l]$ P*-a.s.*

$$\xi(t) = \mathcal{S}\left(\int_0^t \Gamma\bar{\alpha}(\tau,\xi(\tau),\dot{\xi}(\tau))d\tau + \right.$$

$$\left. + \int_0^t \Gamma A(\tau,\xi(\tau),\dot{\xi}(\tau))dw(\tau) + C\right). \qquad (4.3)$$

Definition 4.3. *We say that (4.2) has a strong solution on* $[0,l] \subset R$ *with initial conditions* $\xi(0) = m_0$, $\dot{\xi}(0) = C$ *if on any probability space* $(\Omega,\mathcal{F},\mathrm{P})$, *such that it admits a Wiener process, and for any Wiener process* $w(t)$ *in* $\mathbb{R}^n$, *defined on* $(\Omega,\mathcal{F},\mathrm{P})$, *there exist: a stochastic process* $\xi(t)$ *with a.s.* C^1*-smooth sample paths in* M, *defined on* $(\Omega,\mathcal{F},\mathrm{P})$ *and non-anticipating with respect to* $w(t)$ *with initial condition* $\xi(0) = m_0$ *and* $\dot{\xi}(0) = C$, *a single-valued vector field* $\bar{\alpha}(t,m,X)$ *on* M *and a single-valued* $(1,1)$*-tensor field* $A(t,m,X)$ *such that*

(i) for all t *the random vector* $\bar{\alpha}(t,\xi(t),\dot{\xi}(t))$ *belongs to* $\boldsymbol{\alpha}(t,\xi(t),\dot{\xi}(t))$ P*-a.s.;*

(ii) for all t *the random tensor* $A(t,\xi(t),\dot{\xi}(t))$ *belongs to* $\mathbf{A}(t,\xi(t),\dot{\xi}(t))$ P*-a.s.;*

(iii) the integrals $\int_0^t \Gamma\bar{\alpha}(\tau,\xi(\tau),\dot{\xi}(\tau))d\tau$ *and* $\int_0^t \Gamma A(\tau,\xi(\tau),\dot{\xi}(\tau))dw(\tau)$ *are well-posed for* $\xi(t)$, $w(t)$, $\bar{\alpha}$ *and* A *and* P*-a.s. (4.3) holds for all* $t \in [0,l]$.

As well as in Section 3 one can easily prove that $\xi(t)$ as above satisfies (4.3) if and only if its velocity hodograph $v(t)$ (i.e., $\xi(t) = \mathcal{S}v(t)$) satisfies the velocity hodograph equation of the form

$$v(t) = \int_0^t \Gamma\bar{\alpha}\left(\tau,\mathcal{S}v(\tau),\frac{d}{d\tau}\mathcal{S}v(\tau)\right)d\tau +$$

$$+\int_0^t \Gamma A\left(\tau,\mathcal{S}v(\tau),\frac{d}{d\tau}\mathcal{S}v(\tau)\right)dw(\tau) + C \qquad (4.4)$$

that is an equation of diffusion type in the tangent (i.e., linear) space at m_0 and so it is more convenient for investigation. Below, we shall find $\bar{\alpha}$ and A as in Definitions 4.2 and 4.3 and corresponding $v(t)$, being a solution of (4.4) in weak or strong sense, and then obtain $\xi(t) = \mathcal{S}v(t)$ satisfying (4.3).

If both $\boldsymbol{\alpha}$ and $\mathbf{A}$ have continuous selectors satisfying Itô condition (see (4.5) below), the existence of weak solution trivially follows from that for Langevin equation obtained in Section 3. If it is not the case the existence problem for Langevin inclusions requires special constuctions.

We present the following modification of the notion of ε-approximation for set-valued mappings.

Definition 4.4. *A continuous single-valued force field* $\bar{\alpha}_\varepsilon(t,m,X)$ *is called* ε*-approximation of the set-valued force field* $\boldsymbol{\alpha}(t,m,X)$ *on* M *if its graph* $(t,m,X,\bar{\alpha}_\varepsilon(t,m,X))$ *lies in the* ε*-neighbourhood of* $(t,m,X,\boldsymbol{\alpha}(t,m,X))$ *(the graph of* $\boldsymbol{\alpha}$*) in* $[0,l]\times TM \oplus TM$ *where* $\oplus$ *denotes the Whitney sum. For* $(1,1)$*-tensor fields the definition is analogous.*

In [3] the following statement is proved.

Theorem 4.5. *Let* $\Phi : \mathbb{R}^n \multimap \mathbb{R}^n$ *be an upper semi-continuous set-valued map with convex closed bounded values. For a sequence* $\varepsilon_i \to 0$ *there exists a sequence of continuous* ε_i *approximations for* Φ *that point-wise converges to a Borel measurable selector of* Φ. *If* Φ *takes values in a convex set* Ξ *in* $\mathbb{R}^n$, *those* ε*-approximations take values in* Ξ *as well.*

One can easily see that the natural analogue of Theorem 4.5 holds for both vector and $(1,1)$-tensor force fields on a manifold.

We say that $\boldsymbol{\alpha}$ and $\mathbf{A}$ satisfy the Itô condition if they have linear growth in velocities, i.e., there exists a certain $\Theta > 0$ so that the following inequality:

$$\|\boldsymbol{\alpha}(t,m,X)\| + \|\mathbf{A}(t,m,X)\| < \Theta(1 + \|X\|) \quad (4.5)$$

holds.

Theorem 4.6. *Let the set-valued vector force field $F(t,m,X)$ and set-valued $(1,1)$-tensor force field $A(t,m,X)$ be upper semi-continuous with convex bounded closed values and satisfy Itô condition (4.5) for a certain Θ.*

Then for any $m_0 \in M$ and $C \in T_{m_0}M$ the Langevin inclusion (4.2) has a weak solution with initial conditions $\xi(0) = m_0$, $\dot{\xi}(0) = C$, well-posed for all $t \in [0,\infty)$.

Proof. Specify $l > 0$. Denote by $\mathcal{B}$ the Borel σ-algebra on $[0,l]$ and by λ the normalised Lebesgue measure on it. Consider $\Omega = C^0([0,l], T_{m_0}M)$, the Banach space of continuous curves $x:[0,l] \to T_{m_0}M$ with usual uniform norm, and $\mathcal{F}$, the σ-algebra generated by cylindrical sets on Ω. By $\mathcal{P}_t$ we denote the σ-algebra, generated by cylinder sets with bases over $[0,t]$.

We shall use several measures on $(\Omega,\mathcal{F})$ and on the product space $[0,l]\times\Omega$ we shall introduce the corresponding product measures.

Take a sequence $\varepsilon_i \to 0$ and construct sequences $f_i(t,m,X)$ and $a_i(t,m,X)$ of continuous ε_i-approximations of $F(t,m,X)$ and $A(t,m,X)$, respectively, as in Theorem 4.5. Namely, denote by $\Psi_i(t,m,X)$ a continuous set-valued force field with convex closed values whose graph belongs to the ε_i-neighbourhood of the graph of $F(t,m,X)$ and such that for all (t,m,X) the inclusion $F(t,m,X) \subset \Psi_i(t,m,X)$ holds (the existence of such $\Psi_i(t,m,X)$ follows from [24]). Then as well the minimal selectors $f_i(t,m,X)$ of $\Psi_i(t,m,X)$ point-wise converge to the minimal selector $f(t,m,X)$ of $F(t,m,X)$ as $i\to\infty$ and $f(t,m,X)$ is Borel measurable as a point-wise limit of continuous mappings. In complete analogy with these arguments we introduce a continuous $(1,1)$-tensor field $\hat{\Psi}_i(t,m,X)$ whose graph belongs to the ε_i-neighbourhood of the graph of $A(t,m,X)$ and such that for all (t,m,X) the inclusion $A(t,m,X)\subset\hat{\Psi}_i(t,m,X)$ holds. The minimal selectors $a_i(t,m,X)$ of $\hat{\Psi}_i(t,m,X)$ point-wise converge to the minimal selector $a(t,m,X)$ of $A(t,m,X)$ as $i\to\infty$ and $a(t,m,X)$ is Borel measurable.

Taking into account Definition 4.4 and inequality (4.5) it is evident that

$$\|f_i(t,m,X)\| + \|a_i(t,m,X)\| < Q(1+\|X\|)$$

for a certain $Q > \Theta$ and for all i.

Pass from the sequences $f_i(t,m,X)$ and $a_i(t,m,X)$ to the sequences $\tilde{f}_i:[0,l]\times\Omega\to TM$ and $\tilde{a}_i:[0,l]\times\Omega\to TM$, where $\tilde{f}_i(t,x(\cdot)) = f_i(t,\mathcal{S}x(t),\frac{d}{dt}\mathcal{S}x(t))$ and $\tilde{a}_i(t,x(\cdot)) = a_i(t,\mathcal{S}x(t),\frac{d}{dt}\mathcal{S}x(t))$. Introduce also $\tilde{f}(t,x(\cdot)) = f(t,\mathcal{S}x(t),\frac{d}{dt}\mathcal{S}x(t))$ and $\tilde{a}(t,x(\cdot)) = a(t,\mathcal{S}x(t),\frac{d}{dt}\mathcal{S}x(t))$.

Consider the maps $\Gamma\tilde{f}_i(t,x(\cdot))$ from $[0,l]\times\Omega$ into $T_{m_0}M$ and $\Gamma\tilde{a}_i(t,x(\cdot))$ from $[0,l]\times\Omega$ into linear endomorphisms on $T_{m_0}M$.

Since $\frac{d}{dt}\mathcal{S}x(t)$ is by the construction parallel to $x(t)$ along $\mathcal{S}x(\cdot)$ and the parallel translation preserves the norms, we get

$$\|\Gamma\tilde{f}_i(t,x(\cdot))\| + \|\Gamma\tilde{a}_i(t,x(\cdot))\| < Q(1+\|x(\cdot)\|). \quad (4.6)$$

By the construction, $\Gamma\tilde{f}_i(t,x(\cdot))$ and $\Gamma\tilde{a}_i(t,x(\cdot))$ are continuous on $[0,l]\times\Omega$ (this follows from the continuity of Γ, see [11, 12, 13, 14]) and measurable with respect to the σ-subalgebra $\mathcal{P}_t$ in $\mathcal{F}$ generated by cylindrical sets with bases over $[0,t]$. Since it also satisfies (4.6), there exists a weak solution $v_i(t)$ of the equation

$$v_i(t) = \int_0^t \Gamma\tilde{f}_i(\tau, v_i(\cdot))d\tau + \int_0^t \Gamma\tilde{a}_i(\tau, v_i(\cdot))dw(t) + C \quad (4.7)$$

(see Theorem III.2.4 of [21]). Denote by μ_i the measure on $(\Omega,\mathcal{F})$ corresponding to v_i. Recall that $v_i(t)$ is represented as the coordinate process $v_i(t,x(\cdot)) = x(t)$ on the probability space $(\Omega,\mathcal{F},\mu_i)$.

By routine method (see, e.g., [21]), since all $\Gamma\tilde{f}_i(t,x(\cdot))$ and $\Gamma\tilde{a}_i(t,x(\cdot))$ satisfy (4.6) with the same Q, one can show that the set of measures $\{\mu_i\}$ is weakly compact and so there exists a subsequence converging weakly to a certain probability measure μ on $(\Omega,\mathcal{F})$. For the sake of convenience we do not change the notations and say that μ_i itself is that converging subsequence. Denote by $v(t)$ the coordinate process on $(\Omega,\mathcal{F},\mu)$.

Introduce measures ν_i on $(\Omega,\mathcal{F})$ by the relations $d\nu_i = (1+\|x(\cdot)\|)d\mu_i$. They weakly converge to ν defined by the relation $d\nu = (1+\|x(\cdot)\|)d\mu$ (see, e.g., [3,21]).

As $\Gamma\tilde{f}_k(t,x(\cdot))$ converge to $\Gamma\tilde{f}(t,x(\cdot))$ point-wise, they converge a.s. with respect to all $\lambda\times\mu_i$, and so the functions $\frac{\Gamma\tilde{f}_k(t,x(\cdot))}{1+\|x(\cdot)\|}$ converge to $\frac{\Gamma\tilde{f}(t,x(\cdot))}{1+\|x(\cdot)\|}$ a.s. with respect to all $\lambda\times\nu_i$. Specify $\delta>0$. By Egorov's theorem (see, e.g., [25]) for any i there exists a subset $\tilde{K}^i_\delta \subset [0,l]\times\Omega$ such that $(\lambda\times\nu_i)(\tilde{K}^k_\delta) > 1-\delta$, and the sequence $\frac{f^*_k(t,x(\cdot))}{1+\|x(\cdot)\|}$ converges to $\frac{\tilde{f}(t,x(\cdot))}{1+\|x(\cdot)\|}$

uniformly on $\tilde{K}^i_\delta$. Introduce $(\tilde{K}_\delta = \bigcup_{i=0}^{\infty}\tilde{K}^i_\delta)$. The sequence $\frac{\Gamma\tilde{f}_k(t,x(\cdot))}{1+\|x(\cdot)\|}$ converges to $\frac{\Gamma\tilde{f}(t,x(\cdot))}{1+\|x(\cdot)\|}$ uniformly on $\tilde{K}_\delta$ and $(\lambda\times\nu_i)(\tilde{K}_\delta) > (\lambda\times\nu_i)([0,l]\times\Omega)-\delta$ for all $i=0,\dots,\infty$.

Notice that $\Gamma\tilde{f}(t,x(\cdot))$ is continuous on a set of full measure $\lambda\times\nu$ on $[0,l]\times\Omega$. Indeed, consider a sequence $\delta_i\to 0$ and the corresponding sequence $\tilde{K}_{\delta_i}$ from Egorov's theorem. By the above construction $\Gamma\tilde{f}(t,x(\cdot))$ is a uniform limit of continuous functions on each $\tilde{K}_{\delta_i}$. Thus it is continuous on each $\tilde{K}_{\delta_i}$ and so, on every finite union $\bigcup_{i=1}^{n}\tilde{K}_{\delta_i}$.

Evidently $\lim_{n\to\infty}(\lambda\times\nu)(\bigcup_{i=1}^{n}\tilde{K}_{\delta_i}) = (\lambda\times\nu)([0,l]\times\Omega)$.

Hence $\frac{\Gamma\tilde{f}(t,x(\cdot))}{1+\|x(\cdot)\|}$ is continuous on a set of full measure $\lambda\times\nu$ on $[0,l]\times\Omega$.

Let $g_t(x(\cdot))$ be a bounded (say, $|g_t(x(\cdot))|<\Xi$ for all $x(\cdot)\in\Omega$) and continuous $\mathcal{P}_t$-measurable function on Ω.

Because of the above uniform convergence on $\tilde{K}_\delta$ for all k and boundedness of g_t we get that for k large enough

$$\left\|\int_{\tilde{K}_\delta}\left(\int_t^{t+\Delta t}(\Gamma\tilde{f}_k(\tau,x(\cdot))-\Gamma\tilde{f}(\tau,x(\cdot)))d\tau\right)g_t(x(\cdot))d\mu_k\right\| =$$

$$= \left\|\int_{\tilde{K}_\delta}\left(\int_t^{t+\Delta t}\frac{\Gamma\tilde{f}_k(\tau,x(\cdot))-\Gamma\tilde{f}(\tau,x(\cdot))}{1+\|x(\cdot)\|}d\tau\right)g_t(x(\cdot))d\nu_k\right\| < \delta.$$

Since $(\lambda\times\mu_k)(\tilde{K}_\delta)>1-\delta$ for all k, $\left\|\frac{\Gamma\tilde{f}_k(t,x(\cdot))}{1+\|x(\cdot)\|}\right\|<Q$ for all $k=0,1,\dots,\infty$ (i.e., $\Gamma\tilde{f}$ is included) and $|g_t(x(\cdot))|<\Xi$ (see above), we get

$$\left\|\int_{\Omega\setminus\tilde{K}_\delta}\left(\int_t^{t+\Delta t}(\Gamma\tilde{f}_k(\tau,x(\cdot))-\Gamma\tilde{f}(\tau,x(\cdot)))d\tau\right)g_t(x(\cdot))d\mu_k\right\| =$$

$$= \left\|\int_{\Omega\setminus\tilde{K}_\delta}\left(\int_t^{t+\Delta t}\frac{\Gamma\tilde{f}_k(\tau,x(\cdot))-\Gamma\tilde{f}(\tau,x(\cdot))}{1+\|x(\cdot)\|}d\tau\right)g_t(x(\cdot))d\nu_k\right\| <$$

$$< 2Q\Xi\delta.$$

From the fact that δ is an arbitrary positive number it follows that

$$\lim_{k\to\infty}\int_\Omega\left(\int_t^{t+\Delta t}\Gamma\tilde{f}_k(\tau,x(\cdot))d\tau - \int_t^{t+\Delta t}\Gamma\tilde{f}(\tau,x(\cdot))d\tau\right)g_t(x(\cdot))d\mu_k = 0.$$

The function $\frac{\Gamma\tilde{f}(t,x(\cdot))}{1+\|x(\cdot)\|}$ is $\lambda\times\nu$-a.s. continuous (see above) and bounded on $[0,l]\times\Omega$. Hence by Lemma in section VI.4 of [26] from the weak convergence of ν_k to ν it follows that

$$\lim_{k\to\infty}\int_\Omega\left(\int_t^{t+\Delta t}\Gamma\tilde{f}(\tau,x(\cdot))d\tau\right)g_t(x(\cdot))d\mu_k =$$
$$= \lim_{k\to\infty}\int_\Omega\left(\int_t^{t+\Delta t}\frac{\Gamma\tilde{f}(\tau,x(\cdot))}{1+\|x(\cdot)\|}d\tau\right)g_t(x(\cdot))d\nu_k =$$
$$= \int_\Omega\left(\int_t^{t+\Delta t}\frac{\Gamma\tilde{f}(\tau,x(\cdot))}{1+\|x(\cdot)\|}d\tau\right)g_t(x(\cdot))d\nu =$$
$$= \int_\Omega\left(\int_t^{t+\Delta t}\Gamma\tilde{f}(\tau,x(\cdot))d\tau\right)g_t(x(\cdot))d\mu. \qquad (4.8)$$

Obviously

$$\lim_{i\to\infty}\int_\Omega(x(t+\Delta t)-x(t))d\mu_i =$$
$$= \lim_{i\to\infty}\int_\Omega\frac{x(t+\Delta t)-x(t)}{1+\|x(\cdot)\|}d\nu_i =$$
$$= \int_\Omega\frac{x(t+\Delta t)-x(t)}{1+\|x(\cdot)\|}d\nu = \int_\Omega(x(t+\Delta t)-x(t))d\mu. \qquad (4.9)$$

Notice that

$$\int_\Omega\Big(x(t+\Delta t)-x(t) - \int_t^{t+\Delta t}\Gamma\tilde{f}_k(\tau,x(\cdot))d\tau\Big)g_t(x(\cdot))d\mu_k = 0 \qquad (4.10)$$

since

$$\int_\Omega(x(t+\Delta t)-x(t))g_t(x(\cdot))d\mu_k = E[(v_k(t+\Delta t)-v_k(t))g_t(v_k(t))],$$

$$\int_\Omega\left(\int_t^{t+\Delta t}\Gamma\tilde{f}_k(\tau,x(\cdot))d\tau\right)g_t(x(\cdot))d\mu_k = E\left[\left(\int_t^{t+\Delta t}\Gamma\tilde{f}_k(\tau,v_k(\tau))d\tau\right)g_t(v_k(t))\right]$$

and $v_k(t)$ is a solution of (4.7).

From (4.8), (4.9) and (4.10) it follows that

$$\int_\Omega\Big[(x(t+\Delta t)-x(t)) - \int_t^{t+\Delta t}\Gamma\tilde{f}(s,x(\cdot))ds\Big]g_t(x(\cdot))d\mu = 0. \qquad (4.11)$$

From (4.11) it evidently follows that the process $v(t)-\int_0^t\Gamma\tilde{f}(s,v(s))ds$ is a martingale on $(\Omega,\mathcal{F},\mu)$ with respect to $\mathcal{P}_t$.

Specify a certain orthonormal basis in $T_{m_0}M$. Then the vectors in $T_{m_0}M$ are considered as coordinate columns. If X is such a vector, the transposed row vector is denoted by X^*. Notice that for a column X and a row Y^* the product XY^* with respect to matrix multiplication, is a matrix. Linear operators from $T_{m_0}M$ to $T_{m_0}M$ are represented in coordinates as $n\times n$ matrices, the symbol $*$ means transposition of a matrix (pass to the matrix of conjugate operator).

Consider the sequence $a_i(t,m,X)$ of ε_i-approximations of $A(t,m,X)$, that point-wise converges to the Borel-measurable selector $a(t,m,X)$ (see the beginning of this proof). One can easily see that $a_i(t,m,X)(a_i(t,m,X))^*$ point-wise converges to $a(t,m,X)(a(t,m,X))^*$. Then in complete analogy with the above argument one can show that

$$\int_\Omega [(x(t+\Delta t)-x(t))(x(t+\Delta t)-x(t))^* - \\ -\int_t^{t+\Delta t} \Gamma\tilde{a}(s,x(\cdot))(\Gamma\tilde{a}(s,x(\cdot)))^* ds] g_t(x(\cdot))d\mu = 0 \quad (4.12)$$

with the same g_t as above.

Using standard Girsanov technique one can derive from (4.11) and (4.12) that on $(\Omega,\mathcal{F},\mu)$ there exists a Wiener process $w(t)$, adapted to $\mathcal{P}_t$, such that $v(t)$ on $(\Omega,\mathcal{F},\mu)$ satisfies the equality

$$v(t) = C + \int_0^t \Gamma\tilde{f}(s,v(\cdot))ds + \int_0^t \Gamma\tilde{a}(s,v(\cdot))dw(s) \quad (4.13)$$

(see [21]). Then, taking into account the construction of $\tilde{f}$ and operators $\mathcal{S}$ and Γ, one can easily see that the process $\xi(t) = \mathcal{S}v(t)$ satisfies the equation

$$\xi(t) = \mathcal{S}(\int_0^t \Gamma f(s,\xi(s),\frac{d}{ds}\xi(s))ds + \\ +\int_0^t \Gamma a(s,\xi(s),\frac{d}{ds}\xi(s))dw(s) + C). \quad (4.14)$$

Since $f(t,m,X) \in F(t,m,X)$ and $a(t,m,X) \in A(t,m,X)$ and $l > 0$ is an arbitrary number, this completes the proof. ■

In some cases we can prove existence of strong solution of Langevin inclusion (4.2). Let us present an example of such existence theorem.

In what follows we use $[0,l]$, $\mathcal{B}$, Ω, $\mathcal{F}$ and $\mathcal{P}_t$ introduced in the proof of Theorem 4.6. By $\mathcal{B}_t$ we denote the Borel σ-algebra on $[0,t]$ for $t \in [0,l]$.

Introduce the notation $compZ$ for the space of compact subsets in the metric space Z. Thus, we say that the set-valued vector field $B(t,m,X)$ sends $[0,l]\times TM$ into $compTM$ if for any $(t,m,X) \in [0,l]\times TM$ the image $B(t,m,X) \subset T_mM$ is compact.

Recall several definitions.

Definition 4.7. *A single-valued map* $\beta : [0,l]\times\Omega \to \mathbb{R}^n$ *is called* $\{\mathcal{P}_t\}$*-progressively measurable if for every* t *it is measurable with respect to* $\mathcal{B}_t \times \mathcal{P}_t$.

Definition 4.8. *A set-valued map* $B : [0,l]\times\Omega \to comp\mathbb{R}^n$ *is called* $\{\mathcal{P}_t\}$*-progressively measurable if* $\{(t,\omega) \in [0,l]\times\Omega \mid B(t,\omega)\cap C \neq \varnothing\} \in \mathcal{B}_t \times \mathcal{P}_t$ *for every closed set* $C \subset \mathbb{R}^n$.

Definition 4.9. *We say that a set-valued vector field* $B : [0,l]\times TM \to compTM$

(i) is dissipative if for all $t \in [0,l]$, $m \in M$, $X, Y \in T_mM$ *and* $U \in B(t,m,X)$, $V \in B(t,m,Y)$ *the inequality* $\langle X-Y, U-V\rangle \le 0$ *holds.*

(ii) is maximal if for t, m, X, Y *and* V *as in (i) the inequality* $\langle X-Y, U-V\rangle \le 0$ *is equivalent to the assumption that* $U \in B(t,m,X)$.

Denote by $w(t)$ a certain one-dimensional Wiener process. Let $F(t,m,X)$ and $G(t,m,X)$ be set-valued vector force fields on M as above. Then we can consider the stochastic differential inclusion of Langevin type

$$\xi(t) \in \mathcal{S}(\int_0^t \Gamma F(\tau,\xi(\tau),\dot{\xi}(\tau))d\tau + \\ +\int_0^t \Gamma G(\tau,\xi(\tau),\dot{\xi}(\tau))dw(\tau) + C). \quad (4.15)$$

Inclusion (4.15) is a particular case of (4.2) since $\Gamma G(\tau,\xi(\tau),\dot{\xi}(\tau))dw(\tau)$ can be represented as $\Gamma G(\tau,\xi(\tau),\dot{\xi}(\tau))(PdW(\tau))$ where P is the orthogonal projection onto the linear span of vector $\Gamma G(\tau,\xi(\tau),\dot{\xi}(\tau))$.

Theorem 4.10. *Let the set-valued vector fields* $F(t,m,X)$ *and* $G(t,m,X)$, $F,G : [0,l]\times TM \to compTM$, *be Borel measurable, uniformly bounded, dissipative and maximal. Then there exists a strong solution of (4.15), well posed for* $t \in [0,l]$, *with initial conditions* $\xi(0) = m_0$ *and* $\dot{\xi}(0) = C$ *for any* $m_0 \in M$ *and* $C \in T_{m_0}M$.

Proof. Let $(\tilde{\Omega},\tilde{\mathcal{F}},\mathrm{P})$ be a probability space admitting a one-dimensional Wiener process $w(t)$. Denote by $\mathcal{P}_t^w$ the σ-subalgebra of $\tilde{\mathcal{F}}$ generated by all $w(s)$ for $0 \le s \le t$ and completed by all sets of zero probability. Let $Y : \tilde{\Omega} \to \Omega$ be a measurable map. From the properties of parallel translation and the assumed hypothesis one can easily derive that the coefficients

$$\Gamma F(t,\omega,Y) = \Gamma F(t,\mathcal{S}Y(\omega)(t),\frac{d}{dt}\mathcal{S}Y(\omega)(t))$$

and

$$\Gamma G(t,\omega,Y) = \Gamma G(t,\mathcal{S}Y(\omega)(t),\frac{d}{dt}\mathcal{S}Y(\omega)(t))$$

for $\omega \in \tilde{\Omega}$ satisfy all conditions of Theorem 1 [27] and so on $(\tilde{\Omega},\tilde{\mathcal{F}},\mathrm{P})$ there exists a continuous $\mathcal{P}_t^w$-progressively measurable process $v(t)$

($v(0)=0$) in $T_{m_0}M$ and L^2-selectors $f(t,\omega)$ of $\Gamma F(t,\omega,v)$ and $g(t,\omega)$ of $\Gamma G(t,\omega,v)$ such that a.s.

$$v(t)=\int_0^t f(\tau,\omega)d\tau+\int_0^t g(\tau,\omega))dw(\tau)+C. \quad (4.16)$$

Consider the M-valued process $\xi(t)=\mathcal{S}v(t)$ with $v(t)$ satisfying (4.16). In the same manner as in the proof of Theorem 4.6 we can construct Borel measurable selectors $f(t,m,X)$ of $F(t,m,X)$ and $g(t,m,X)$ of $G(t,m,X)$ such that a.s.

$$\xi(t)=\mathcal{S}(\int_0^t \Gamma f(\tau,\xi(\tau),\dot{\xi}(\tau))d\tau+$$

$$+\int_0^t \Gamma g(\tau,\xi(\tau),\dot{\xi}(\tau))dw(\tau)+C).$$

■

5. SYSTEMS WITH RANDOM PERTURBATION OF VELOCITY

In previous two sections we dealt with equations obtained from the ordinary Newton's law by a stochastic perturbation of vertical component of right-hand side, i.e., of the force field (see the Introduction). Here we investigate the systems, in which the horizontal part is subjected to stochastic influence. This means that a random perturbation of velocity arises. Such a situation can appear, e.g., if a particle, subjected to a deterministic force, moves in addition with a random media. Note that in this model example the perturbation would be independent of the particle velocity.

Taking into account the above model example, we investigate system (1.1) with $\bar{\alpha}$ independent of velocity, i.e., it turns into

$$\begin{cases}\dot{x}(t)=v(t)\\ \dot{v}(t)=\bar{\alpha}(t,x(t)).\end{cases} \quad (5.1)$$

A particular example of such a force is $-grad\,\mathcal{U}$ in a conservative mechanical system where $\mathcal{U}$ is the potential energy.

Now suppose that in system (5.1) the right-hand side of horizontal (i.e. the first) equation is subjected to random perturbation of the form $A(t,x(t))\dot{w}(t)$ where $\dot{w}(t)$ is white noise. Note that this perturbation is independent of velocity of the particle. In appropriate terms this means that the process $\xi(t)$ describing the motion of particle, satisfies the equality $\xi(t)=\xi_0+\int_0^t v(s,\xi(s))ds+$ $+\int_0^t A(s,\xi(s))dw(s)$ where the vector field $v(t,x)$ satisfies the relation $Dv(t,\xi(t))=\bar{\alpha}(t,\xi(t))$. The formal equation of motion in terms of forward mean derivatives then takes the form

$$\begin{cases}D\xi(t)=v(t,\xi(t))\\ D_2\xi(t)=A(t,\xi(t))A^*(t,\xi(t))\\ Dv(t,\xi(t))=\bar{\alpha}(t,\xi(t))\end{cases} \quad (5.2)$$

where $Dv(t,\xi(t))$ is given by formula (2.6).

We also suppose that $A(t,x)$ and $\bar{\alpha}(t,x))$ satisfy the Itô condition

$$\|A(t,x)\|+\|\bar{\alpha}(t,x)\|<K(1+\|x\|) \quad (5.3)$$

for some $K>0$.

Theorem 5.1. *Let $A(t,x)$ and $\bar{\alpha}(t,x)$ be jointly continuous in t,x and satisfy (5.3). Then for every couple ξ_0, $v_0\in\mathbb{R}^n$ there exists a weak solution of (5.2) with initial conditions $\xi(0)=\xi_0$ and $v(0)=v_0$.*

Proof. In $C^0([0,l],\mathbb{R}^n)$ introduce the σ-algebra $\tilde{\mathcal{F}}$ generated by cylindrical sets. By $\tilde{\mathcal{P}}_t$ denote the σ-algebra generated by cylindrical sets over $[0,t]\subset[0,l]$.

Consider the map $\bar{v}:[0,l]\times C^0([0,l],\mathbb{R}^n)\to\mathbb{R}^n$ defined by the formula

$$\bar{v}(t,x(\cdot))=v_0+\int_0^t \bar{\alpha}(\tau,x(\cdot)))d\tau. \quad (5.4)$$

By the construction this map is continuous jointly in $t\in[0,l]$ and $x(\cdot)\in C^0([0,l],\mathbb{R}^n)$. In addition it is obvious that if $x_1(\cdot)$ and $x_2(\cdot)$ coincide on $[0,t]$ then $\bar{v}(t,x_1(\cdot))=\bar{v}(t,x_2(\cdot))$. This means that $\bar{v}(t,x(\cdot))$ is measurable with respect to $\tilde{\mathcal{P}}_t$. (see, e.g., [21]).

Taking into account (5.3) one can easily derive the inequality

$$\|\bar{v}(t,x(\cdot))\|=\left\|\int_0^t \bar{\alpha}(\tau,x(\cdot)))d\tau\right\|\le$$

$$\le\int_0^t \|\bar{\alpha}(\tau,x(\cdot)))\|d\tau\le K\int_0^t(1+\|x(\tau)\|)d\tau\le$$

$$\le K\int_0^t(1+\|x(\cdot)\|_{C^0})ds\le lK(1+\|x(\cdot)\|_{C^0})$$

where $\|\cdot\|_{C^0}$ is the norm in $C^0([0,l],\mathbb{R}^n)$.

Introduce $A(t,x(\cdot))$ as $A(t,x(\cdot))=A(t,x(t))$. Notice that $A(t,x(\cdot))$ is measurable with respect to $\tilde{\mathcal{P}}_t$ and that from (5.3) it follows that $\|A(t,x(\cdot))\|\le K(1+\|x(\cdot)\|_{C^0})$. So, both $\bar{v}(t,x(\cdot))$ and $A(t,x(\cdot))$ satisfy the Itô condition in the form

$$\|\bar{v}(t,x(\cdot)\|+\|A(t,x(\cdot)\|\le\bar{K}(1+\|x(\cdot)\|_{C^0})$$

with $\bar{K}=max(K,lK)$.

Now the couple $\bar{v}(t,x(\cdot))$ and $A(t,x(\cdot))$ satisfies all conditions of theorem III.2.4 from [21], hence, the stochastic differential equation

$$x(t) = x_0 + \int_0^t \overline{v}(s, x(\cdot))ds + \int_0^t A(s, x(\cdot))dw(s) \quad (5.5)$$

has a weak solution on $[0, l]$. This means that there exist a probabilistic measure μ on $(C^0([0,l],\mathbb{R}^n), \mathcal{F})$ and a Wiener process in $\mathbb{R}^n$, given on $(C^0([0,l],\mathbb{R}^n),\mathcal{F},\mu)$ and adapted to $\mathcal{P}_t$, such that the coordinate process $x(t)$ on $(C^0([0,l],\mathbb{R}^n),\mathcal{F},\mu)$ and $w(t)$ satisfy (5.5). Introduce $v(t,x)A$ as the regression $v(t,x) = E(\overline{v}(t,x(\cdot)) \mid x(t) = x)$. This together with construction of process $\overline{v}(t,x(\cdot))$ completes the proof of Theorem. ■

The simple construction used in proof of Theorem 5.1, can be generalized to be applicable in more complicated situation. First we consider the case where the force field is set-valued, lower semi-continuous but not necessarily has convex values.

Let $F(t,x)$ be a lower semi-continuous set-valued map $F : \mathbb{R}\times\mathbb{R}^n \multimap \mathbb{R}^n$ with closed images and $A(t,x) : \mathbb{R}^n \to \mathbb{R}^n$ be a field of single-valued linear operators jointly continuous in parameters $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. We suppose that $F(t,x)$ and $A(t,x)$ satisfy Itô condition, i.e., that there exists a constant $\Theta > 0$ such that

$$\|F(t,x)\| + \|A(t,x)\| < \Theta(1 + \|x\|) \quad (5.6)$$

for all $t \in R$ and $x \in \mathbb{R}^n$ where $\|A(t,x)\|$ is the operator norm and $\|F(t,x)\| = \sup_{y\in F(t,x)} \|y\|$.

Equation (system) (5.3) now is replaced by the following inclusion

$$\begin{cases} D\xi(t) = v(t,\xi(t)) \\ D_2\xi(t) = A(t,\xi(t))A^*(t,\xi(t)) \\ Dv(t,\xi(t)) \in F(t,\xi(t)). \end{cases} \quad (5.7)$$

In what follows we consider $\mathbb{R}^n$ and $\mathbb{R}$ with their Borel σ-algebras $\mathcal{B}^n$ and $\mathcal{B}$, respectively. Let $x(\cdot)$ be a continuous curve. Consider the set-valued vector field $F(t,x(t))$ along $x(\cdot)$ and denote by $\mathcal{P}F(\cdot,x(\cdot))$ the set of all measurable selectors of $F(t,x(t))$, i.e., the set of measurable maps $\{f : \mathbb{R}\to\mathbb{R}^n : f(x(t)) \in F(t,x(t))\}$. It is obvious that since condition (5.6) is satisfied, all those selectors are integrable on any finite interval in $\mathbb{R}$ with respect to Lebesgue measure. Denote by $\int\mathcal{P}F(\cdot,x(\cdot))$ the set of integrals with varying upper limits of those selectors.

Recall some facts and notions involved in further considerations. Specify $l > 0$. In what follows we denote by λ the normalized Lebesgue measure on $[0,l]$, i.e., such that $\lambda([0,l]) = 1$.

Lemma 5.2. *Let (Ξ, d) be a separable metric space, X be a Banach space. Consider the space $Y = L^1(([0,l],\mathcal{B},\lambda), X))$ of integrable maps from $[0,l]$ into X. If a set-valued map $G : \Xi \to Y$ is lower semicontinuous and has closed decomposable images, it has a continuous selector.*

This is a particular case of Bressan—Colombo Theorem (see, e.g., [28, 29]).

Denote by $C^0([0,l],\mathbb{R}^n)$ the Banach space of continuous maps from $[0,l]$ to $\mathbb{R}^n$ (i.e., continuous curves in $\mathbb{R}^n$, given on $[0,l]$).

Theorem 5.3. *Let, as mentioned above, $F(t,x)$ be a lower semi-continuous set-valued map $F : \mathbb{R}\times\mathbb{R}^n \multimap \mathbb{R}^n$ with closed values and $A(t,x) : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ be a field of single-valued linear operators jointly continuous in parameters $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Let also (5.6) be fulfilled. Then for any specified $l > 0$, $x_0, v_0 \in \mathbb{R}^n$ inclusion (5.7) has a solution on $[0,l]$ with initial position x_0 and initial velocity v_0.*

Proof. In $C^0([0,l],\mathbb{R}^n)$ introduce the σ-algebra $\tilde{\mathcal{F}}$ generated by cylindrical sets. By $\tilde{\mathcal{P}}_t$ denote the σ-algebra generated by cylindrical sets over $[0,t] \subset [0,l]$.

Consider the set-valued maping B sending $x(\cdot) \in C^0([0,l],\mathbb{R}^n)$ into $\mathcal{P}F(\cdot,x(\cdot))$. Since under condition (5.6) all selectors from $\mathcal{P}F(\cdot,x(\cdot))$ are integrable (see above), B takes values in the space $L^1(([0,l],\mathcal{B},\lambda),\mathbb{R}^n)$. It is known (see, e.g., Section 5.5 from [29]) that under the above-mentioned conditions $B : C^0([0,l],\mathbb{R}^n) \to L^1(([0,l],\mathcal{B},\lambda),\mathbb{R}^n)$ is lower semicontinuous and for any $x(\cdot) \in C^0([0,l],\mathbb{R}^n)$ the set $\mathcal{P}F(\cdot,x(\cdot))$, i.e., the image $B(x(\cdot))$ is decomposable and closed. Thus, by Lemma 5.2 B has a continuous selector $b : C^0([0,l],\mathbb{R}^n) \to b : C^0([0,l],\mathbb{R}^n) \to L^1(([0,l],\mathcal{B},\lambda),\mathbb{R}^n)$.

For any $t \in [0,l]$ introduce the map $f_t : C^0([0,l],\mathbb{R}^n) \to C^0([0,l],\mathbb{R}^n)$ that sends a curve $x(\cdot) \in C^0([0,l],\mathbb{R}^n)$ into the curve

$$f_t(\tau, x(\cdot)) = \begin{cases} x(\tau) \text{ for } \tau \in [0,t] \\ x(t) \text{ for } \tau \in [t,l] \end{cases}.$$

Obviously the map f_t is continuous. Since $f_t(\tau,x(\cdot))$ belongs to $C^0([0,l],\mathbb{R}^n)$, the curve $b(f_t(\tau,x(\cdot))) \in L^1(([0,l],\mathcal{B},\lambda),\mathbb{R}^n)$ is well-posed. By construction $b(f_t(\tau,x(\cdot))) \in F(\tau,x(\tau))$ for almost all $\tau \in [0,t]$ and this selector continuously depends on t in $L^1(([0,l],\mathcal{B},\lambda),\mathbb{R}^n)$.

Consider the map $v : [0,l]\times C^0([0,l],\mathbb{R}^n) \to \mathbb{R}^n$ defined by the formula

$$v(t,x(\cdot)) = v_0 + \int_0^t b(f_t(\tau,x(\cdot)))d\tau. \quad (5.8)$$

By construction this map is continuous jointly in $t \in [0,l]$ and $x(\cdot) \in C^0([0,l],\mathbb{R}^n)$. In addition it is obvious that if $x_1(\cdot)$ and $x_2(\cdot)$ coincide on $[0,t]$ then

$v(t,x_1(\cdot)) = v(t,x_2(\cdot))$. This means that $v(t,x(\cdot))$ is measurable with respect to $\tilde{\mathcal{P}}_t$. (see, e.g., [21]).

Taking into account (5.6) one can easily derive the inequality

$$\|v(t,x(\cdot))\| = \left\|\int_0^t b(f_t(\tau,x(\cdot)))d\tau\right\| \le$$
$$\le \int_0^t \|b(f_t(\tau,x(\cdot)))\|d\tau \le \int_0^t \|F(\tau,x(\tau))\|d\tau \le$$
$$\le \Theta\int_0^t (1+\|x(\tau)\|)d\tau \le \Theta\int_0^t (1+\|x(\cdot)\|_{C^0})ds \le$$
$$\le l\Theta(1+\|x(\cdot)\|_{C^0})$$

where $\|\cdot\|_{C^0}$ is the norm in $C^0([0,l],\mathbb{R}^n)$.

Introduce $A(t,x(\cdot))$ as $A(t,x(\cdot)) = A(t,x(t))$. Notice that $A(t,x(\cdot))$ is measurable with respect to $\tilde{\mathcal{P}}_t$ and that from (5.6) it follows that $\|A(t,x(\cdot))\| \le \le \Theta(1+\|x(\cdot)\|_{C^0})$. So, both $v(t,x(\cdot))$ and $A(t,x(\cdot))$ satisfy the Itô condition in the form

$$\|v(t,x(\cdot)\| + \|A(t,x(\cdot)\| \le \bar{\Theta}(1+\|x(\cdot)\|_{C^0})$$

with $\bar{\Theta} = max(\Theta, l\Theta)$.

Now the couple $v(t,x(\cdot))$ and $A(t,x(\cdot))$ satisfies all conditions of theorem III.2.4 from [21], hence, the stochastic differential equation

$$x(t) = x_0 + \int_0^t v(s,x(\cdot))ds + \int_0^t A(s,x(\cdot))dw(s) \quad (5.9)$$

has a weak solution on $[0,l]$. This means that there exist a probabilistic measure μ on $(C^0([0,l],\mathbb{R}^n)$, $\mathcal{F})$ and a Wiener process in $\mathbb{R}^n$, given on $(C^0([0,l],\mathbb{R}^n),\mathcal{F},\mu)$ and adapted to $\mathcal{P}_t$, such that the coordinate process $x(t)$ on $(C^0([0,l],\mathbb{R}^n),\mathcal{F},\mu)$ and $w(t)$ satisfy (5.9). Taking into account (5.8) and (5.9), one can easily check that this solution satisfies (5.7). ■

For investigating such a problem on manifold the assumptions are more restrictive than in the case of Euclidean space.

Let M be a stochastically complete Riemannian manifold (see, e.g., [13, 14]), on which a certain vector force filed $\bar{\alpha}(t,m)$ independent of velocities, is given. Thus the Newton's law of the mechanical system takes the form

$$\frac{D}{dt}\dot{m}(t) = \bar{\alpha}(t,m(t)).$$

We suppose that the random perturbation of velocity takes the form $A(m)\dot{w}(t)$ where $A(m):\mathbb{R}^k \to \to T_mM$ is a smooth field of linear operators sending a certain Euclidean space $\mathbb{R}^k$ to the tangent spaces to M. We suppose in addition that $A(m)A^*(m) = I$ where I is the unit operator in T_mM. This assumption can be interpreted as the fact that the Riemannian metric on M is determined by diffusion coefficient generated by $A(m)$. In particular it means that we can apply the machinery of equations with unit diffusion coefficient on manifolds from [13, 14].

The equation of motion for the system with random perturbation of velocities is given here in terms of covariant mean derivative $\mathbf{D}$ on manifold introduced in the same manner as in formula (3.4) but with the use of parallel translation along stochastic processes (see,e.g., [13, 14]). We keep the notation $\Gamma_{t,s}$ for operator of such stochastic parallel translation and so (3.4) can be considered as definition of $\mathbf{D}$. As well as above we denote $\Gamma_{0,s}$ by Γ. Taking into account this modification of mean derivatives we introduce the equation of motion for the above system in the form

$$\begin{cases} D\xi(t) = v(t,\xi(t)) \\ D_2\xi(t) = I \\ \mathbf{D}v(t,\xi(t)) = \bar{\alpha}(t,\xi(t)). \end{cases} \quad (5.10)$$

Theorem 5.4. *Let the force field $\bar{\alpha}(t,m)$ be jointly continuous in t,m and uniformly bounded, i.e., $\|\bar{\alpha}(t,m)\| < K$ for all $m \in M$ and $t \in [0,l] \subset \mathbb{R}$ and some $K > 0$. Then for every couple $m_0 \in M$, $v_0 \in T_{m_0}M$ there exists a solution of (5.10) with initial conditions $\xi(0) = m_0$, $v(0) = v_0$ that is well defined on the entire interval $t \in [0,l]$.*

Proof. The idea of proof is analogous to that for Langevin equations. We reduce (5.10) to equation of velocity hodograph type in a single linear space. Then we show that the latter has a weak solution and that its Itô development (see [13, 14]) satisfies (5.10). The difference is that here we use the velocity hodograph equation in terms of stochastic parallel translation (unlike the case of Langevin equation where the ordinary parallel translation was applied).

Consider the space $\tilde{\Omega} = C^0([0,l],T_{m_0}M)$ with the σ-algebra $\tilde{\mathcal{F}}$ generated by cylinder sets and Wiener measure ν on it. On the probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\nu)$ the coordinate process $\tilde{w}(t,x(\cdot)) = x(t)$ is a Wiener process adapted to the family of σ-subalgebras $\mathcal{P}_t$ that for each specified t is generated by cylinder sets with bases on $[0,t]$ and completed by all sets with ν-measure zero.

Since M is stochastically complete, the Itô development $R_I\tilde{w}(t)$, i.e., the Wiener process on M, is well defined on the entire $[0,l]$ for ν-a.s. all curves in $\tilde{\Omega}$ and the parallel translation along ν-almost all sample paths of $R_I\tilde{w}(t)$ is also well-posed (see [13, 14]). Thus we can apply the operator Γ of parallel translation along $R_I\tilde{w}(\cdot)$ from $R_I\tilde{w}(t)$ to $R_I\tilde{w}(0) = m_0$.

Introduce the process $\beta(t, x(\cdot)) = \int_0^t \Gamma\overline{\alpha}(s, R_I x(s))ds$ in $T_{m_0}M$. From the properties of parallel translation and of R_I it follows that $\beta(t)$ is uniformly bounded by the constant lK and that it is non-anticipative with respect to $\mathcal{P}_t$. In addition the density

$$\rho(x(\cdot)) = \exp\left(\int_0^l \langle \beta(t, x(\cdot)), d\tilde{w}(t)\rangle - \frac{1}{2}\int_0^l \beta(t, x(\cdot))^2 dt\right). \quad (5.11)$$

satisfies the equality

$$\int_{\tilde{\Omega}} \rho\, d\nu = 1, \quad (5.12)$$

and so the measure μ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$, introduced by the relation $d\mu = \rho\nu$ is a probability measure.

Then the coordinate process $\overline{v}(t)$ on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \mu)$ satisfies the equation

$$\overline{v}(t) = v_0 + \int_0^t \beta(s, \overline{v}(s))ds + w(s) \quad (5.13)$$

where $w(t)$ is a Wiener process in $T_{m_0}M$, given on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \mu)$ that is non-anticipative with respect to $\mathcal{P}_t$. The Itô development $\xi(t) = R_I\overline{v}(t)$ has the same sample paths as $R_I\tilde{w}(t)$ and so it is well-defined on the entire interval $[0, l]$.

Introduce the vector field $v(t, m)$ as the regression $v(t, x) = E(\Gamma_{0,t}\beta(t) \mid R_I\overline{v}(t) = x)$ Where $\Gamma_{t,0}$ is the operator of parallel translation along $R_I\overline{v}(\cdot)$ from $R_I\overline{v}(0) = m_0$ to $R_I\overline{v}(t)$. Taking into account the properties of Itô development and of parallel translation as well as the construction of covariant mean derivatives, one can easily show that $\xi(t)$ and $v(t, m)$ satisfy (5.10). ■

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