BIFURCATION SOLUTIONS OF BOUNDARY VALUE PROBLEM

M. A. Abdul Hussain

University of Basrah, Basrah, Iraq

This paper studies the bifurcation solutions of boundary value problem . In some domains of parameters, the existence and stability solutions of a certain boundary value problem was shown in which the number of solutions is fixed in every domain.

INTRODUCTION

It is known that many of the nonlinear problems in mathematics and physics can be written in the form of operator equation,

$$f(x,\lambda) = b,\ \ x \in O \subset X,\ \ b \in Y, \lambda \in R^n. \quad (1)$$

in which f is a smooth Fredholm map of index zero, X, Y Banach spaces and O open subset of X. For these problems, the method of reduction to finite dimensional equation [1],

$$\Theta(\xi,\lambda) = \beta,\ \ \xi \in M,\ \ \beta \in N, \quad (2)$$

can be used, where M and N are smooth finite dimensional manifolds.

Passage from equation (1) into equation (2) (variant local scheme of Lyapunov—Schmidt) with the conditions, that equation (2) has all the topological and analytical properties of equation (1) (multiplicity, bifurcation diagram, *etc*...) dealing with [4, 8, 10, 11].

In this work it was assume that $f:\Omega \to F$ is a nonlinear Fredholm map of index zero. A smooth map $f:\Omega \to F$ has variational property, if there exist functional $V:\Omega \to R$ such that $f = \operatorname{grad}_H V$ or equivalently,

$$\frac{\partial V}{\partial x}(x)h = \langle f(x), h\rangle_H,\ \forall x \in \Omega, h \in E.$$

where ($<\cdot,\cdot>_H$ is the scalar product in Hilbert space H).

In this case the solutions of equation $f(x,\lambda) = 0$ are the critical points of functional $V(x,\lambda)$. Suppose that $f:E \to F$ is a smooth Fredholm map of index zero, E, F are Banach spaces and

$$\frac{\partial V}{\partial x}(x,\lambda)h = \langle f(x,\lambda), h\rangle_H,\ \ h \in E,$$

where V is a smooth functional on E. Also it was assume that $E \subset F \subset H$, H is a Hilbert space, then by using method of finite dimensional reduction (Local scheme of Lyapunov—Schmidt) the problem,

$$V(x,\lambda) \to extr,\ \ x \in E,\ \ \lambda \in R^n.$$

can be reduce into equivalent problem,

$$W(\xi,\lambda) \to extr,\ \ \xi \in R^n.$$

the function $W(\xi,\lambda)$ is called Key function.

If $N = span\{e_1,\dots,e_n\}$ is a subspace of E, where $e_1,\dots,e_n$ are orthonormal basis, then Key function $W(\xi,\lambda)$ can be defined in the form,

$$W(\xi,\lambda) = \inf_{x:\langle x,e_i\rangle=\xi_i \forall i} V(x,\lambda),$$

$$\xi = (\xi_1,\dots,\xi_n)^T.$$

Function W has all the topological and analytical properties of functional V (multiplicity, bifurcation diagram, *etc*...) [9]. The study of bifurcation solutions of functional V is equivalent to the study of bifurcation solutions of Key function. If f has variational property, then it is easy to check that,

$$\theta(\xi,\lambda) = \operatorname{grad} W(\xi,\lambda).$$

Equation $\theta(\xi,\lambda) = 0$ is called bifurcation equation. The set of all λ in which function $W(\xi,\lambda)$ has degenerate critical points, is called *Caustic*.

The oscillations and motion of waves of the elastic beams on the elastic support which can be described by means of the following PDE,

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + \alpha\frac{\partial^2 w}{\partial x^2} + \beta w + w^3 = \psi,$$

has been studied by Thompson J M. T., Stewart H. B. [6], Bardin B., Furta S. [2, 3] and another peoples, where w is the deflection of beam and $\psi = \varepsilon\varphi$ (ε — small parameter) is a symmetric function with respect to the involution $I:\psi(x) \mapsto \psi(\pi - x)$. It is known that, to study the oscillations of beams, stationary state should be monitored which is describes by the equation,

$$\frac{d^4 w}{dx^4} + \alpha\frac{d^2 w}{dx^2} + \beta w + w^3 = \psi, \quad (3)$$

In this work equation (3) has been studied with the following conditions,

$$w(0) = w(\pi) = w''(0) = w''(\pi) = 0.$$

The goal of this paper is to study the bifurcation solutions of boundary value problem,

$$\frac{d^4w}{dx^4}+\alpha\frac{d^2w}{dx^2}+\beta w+w^3=\psi, \qquad (4)$$
$$w(0)=w(\pi)=w''(0)=w''(\pi)=0.$$

by using local scheme of Lyapunov—Schmidt to reduce into finite dimensional spaces. Bifurcation solutions of equation (4) with $\varepsilon = 0$, also has been studied by Sapronov Yu. I. [9], he used the method of finite dimensional reduction to solve this problem. In this work the same manner used to solve equation (4) when $\varepsilon \neq 0$.

ANALYSIS OF BIFURCATION

Suppose that $f: E \to F$ is a nonlinear Fredholm operator of index zero from Banach space E in Banach space F, where $E = C^4([0,\pi], R)$ is the space of all continuous functions which have differential of order at most four, $F = C([0,\pi], R)$ is the space of all continuous functions and f can be written in the form of operator equation:

$$f(w,\lambda):=\frac{d^4w}{dx^4}+\alpha\frac{d^2w}{dx^2}+\beta w+w^3, \qquad (5)$$

where $w = w(x)$, Every solution of boundary value problem (4) is a solution of operator equation,

$$f(w,\lambda)=\psi,\ \ \psi\in F. \qquad (6)$$

The purpose is to study bifurcation solutions of equation (6) near the critical point when the dimension of null space is equal two. Since, operator f has variational property, so there exist functional V such that

$$f(w,\lambda)=\operatorname{grad} V(w,\lambda,0).$$

and then every solution of equation (6) is a critical point of functional V where,

$$V(w,\lambda,\psi)=\int_0^{\pi}\left(\frac{(w'')^2}{2}-\alpha\frac{(w')^2}{2}+\right.$$
$$\left.+\beta\frac{w^2}{2}+\frac{w^4}{4}-w\psi\right)dx.$$

Thus, the study of boundary value problem (4) is equivalent to the study extremal problem,

$$V(w,\lambda,\psi)\to extr,\ \ w\in E.$$

Analysis bifurcation can be find by using method of Lyapunov—Schmidt to reduce into finite dimensional space and by localization parameters,

$$\alpha=\alpha_1+\delta_1,\ \ \beta=\beta_1+\delta_2.$$

this reduction lead to the function in two variables,

$$W(\xi,\delta)=\inf_{\langle w,e_i\rangle=\xi_i,i=1,2}V(w,\delta),$$
$$\xi=(\xi_1,\xi_2),\ \ \delta=(\delta_1,\delta_2).$$

It is well known that in the reduction of Lyapunov—Schmidt function $W(\xi,\delta)$ is smooth. This function has all the topological and analytical properties of functional V [9]. In particular, for small δ there is one-to-one corresponding between the critical points of functional V and smooth function W, preserving the type of critical points (multiplicity, index Morse, *etc*...) [9]. Functional V is even, $V(-w,\lambda,0)=V(w,\lambda,0)$ and symmetric with respect to the involution $I: w(x)\mapsto w(\pi - x)$. By using scheme of Lyapunov—Schmidt, the linearized equation corresponding to the equation (6) has the form:

$$h''''+\alpha h''+\beta h=0,\ \ h\in E,$$
$$h(0)=h(\pi)=h''(0)=h''(\pi)=0.$$

This equation give in the $\alpha\beta$-plane characteristic lines. The point of characteristic lines are the points of (α,β) in which equation (4) has non-zero solutions. The point of intersection of characteristic lines in the $\alpha\beta$-plane is a bifurcation point [9]. The result of this intersection lead to bifurcation along the modes $e_1=c_1\sin x,\ e_2=c_2\sin 2x.$ For the boundary value problem (4) the point $(\alpha,\beta)=(5,4)$ is a bifurcation point [9]. Localization parameters,

$$\tilde\alpha=5+\delta_1,\ \ \tilde\beta=4+\delta_2.$$

give rise to the bifurcation along the modes e_1, e_2, where $\|e_1\|=\|e_2\|=1$ and $c_1=c_2=\sqrt{\frac{2}{\pi}}.$

Every vector $w\in E$ can be written in the form,

$$w=u+v,\ \ u=\xi_1e_1+\xi_2e_2\in N,$$
$$v\in E^{\infty-2}=N^{\perp}\cap E.$$

Here $N=Span\{e_1,e_2\}$. By implicit function theorem, there exist smooth map $\Phi: N\to E^{\infty-2}$, such that

$$W(\xi,\delta,\psi)=V(\Phi(\xi,\delta,\psi),\delta,\psi),$$
$$\delta=(\delta_1,\delta_2).$$

and then Key function can be written in the form,

$$W(\xi,\delta)=V(\xi_1e_1+\xi_2e_2+\Phi(\xi_1e_1+\xi_2e_2,\delta),\delta)$$
$$=U(\xi,\delta)+o(|\xi|^4)+O(|\xi|^4)O(\delta).$$

where,

$$U(\xi,\delta)=\frac{1}{4}(\xi_1^4+4\xi_1^2\xi_2^2+\xi_2^4)+$$
$$+\frac{1}{2}(\lambda_1\xi_1^2+\lambda_2\xi_2^2)+q_1\xi_1+q_2\xi_2.$$

Then asymptotic bifurcation of critical points of the function $W(\xi,\delta)$ completely determinate its principal part $U(\xi,\delta)$ [9]. Critical points of the function $U(\xi,\delta)$ are the solutions of the system,

$$\begin{aligned}&\xi_1^3+2\xi_1\xi_2^2+\lambda_1\xi_1+q_1=0,\\&\xi_2^3+2\xi_1^2\xi_2+\lambda_2\xi_2+q_2=0.\end{aligned}\qquad(7)$$

In many applications the Discriminant set (Caustic) can be solve by finding a relationship between the parameters and variables given in the problem, but in some problems there is a difficult for finding this parameterization. The second way for finding the Discriminant set it can by finding the parameter equation, that is; equation of the form

$$h(\delta)=0,\ \ \delta=(\lambda_1,\lambda_2,\ldots,\lambda_n)\in R^n.$$

where $h:R^n\to R$ is a map and $\lambda_1,\lambda_2,\ldots,\lambda_n$ are parameters. Thus, to determine the Discriminant set of the function $U(\xi,\delta)$ it is convenient to find the parameter equation of the form,

$$h_1(\lambda_1,\lambda_2,q_1,q_2)=0,$$
$$h_1:R^4\to R \text{ is a map.}$$

such that the set of all $\lambda=(\lambda_1,\lambda_2,q_1,q_2)$ in which the function $U(\xi,\delta)$ has degenerate critical points will be satisfying the equation $h_1(\lambda)=0$. Since, the function $\psi(x)$ is symmetric with respect to the involution $\psi(x)\mapsto\psi(\pi-x)$ therefore $q_2=0$ and then we have the following result.

Theorem 1. Caustic (bifurcation diagram) of the function $U(\xi,\delta)$ in the space of parameters $(q_1,\lambda_1,\lambda_2)$ is a union of the following three surfaces,

1) $8q_1^2+\lambda_2(\lambda_2-2\lambda_1)^2=0$,
2) $81q_1^2+4(2\lambda_2-\lambda_1)^3=0$,
3) $y-rz=0$,

where

$$y=(108q_1^3-9q_1\lambda_1\lambda_2t+3q_1\lambda_1t^2+q_1t^3)^2,$$
$$r=\frac{1}{4}\left(36q_1^2-\lambda_1\lambda_2t+\frac{\lambda_1}{2}t^2\right)^2,$$
$$z=(36q_1^2-4\lambda_1\lambda_2t),\ \ t=3\lambda_2-4\lambda_1.$$

Proof: The surfaces can be find by solving the following three systems in terms of q_1,λ_1,λ_2. The systems are,

$$\begin{aligned}&\xi_2=0,\\&\xi_1^3+2\xi_1\xi_2^2+\lambda_1\xi_1+q_1=0,\\&(3\xi_1^2+2\xi_2^2+\lambda_1)\times\\&\times(3\xi_2^2+2\xi_1^2+\lambda_2)-16\xi_1^2\xi_2^2=0.\end{aligned}\qquad(8)$$

$$\begin{aligned}&\xi_2=0,\\&\xi_1^3+2\xi_1\xi_2^2+\lambda_1\xi_1+q_1=0,\\&\xi_2^2+2\xi_1^2+\lambda_2=0.\end{aligned}\qquad(9)$$

$$\begin{aligned}&2\xi_2^2-5\xi_1^2+\lambda_1=0,\\&\xi_2^2+2\xi_1^2+\lambda_2=0,\\&\xi_1^3+2\xi_1\xi_2^2+\lambda_1\xi_1+q_1=0.\end{aligned}\qquad(10)$$

system (8) give rise to the quadratic equation of the form $\xi_1^2-a\xi_1+b=0$, where $a=\frac{6q_1}{t}$ and $b=\frac{\lambda_1\lambda_2}{t}$, $t\neq0$. Solve this equation in term of ξ_1 and then substitute the result in the second equation of system (8) we have surface given in (3). From system (9) $\xi_1=\frac{2q_1}{\lambda_2-2\lambda_1}$ and then substitute in the last equation we have surface given in (1). Similarly, system (10) give rise to the surface given in (2). □

To find caustic of equation (6) it is convenient to fixed the value of λ_1 and then find all sections of caustic in the $q_1\lambda_2$-plane as λ_1 change. Thus, we described caustic of equation (6) in the $q_1\lambda_2$-plane for some values of λ_1 with the number and type of critical points in the following Figures 1—3(all figures was found by using Maple 9.5).

Figures (1) , (2) and (3) shows the existence and stability solutions of equation (4). In figure (1) there is only one cusp point in the plane of parameters and the number of critical points is either one minimum or two symmetric minimum and one saddle in every domain of parameters. In figure (2) the number of cusp points in the plane of parameters is three and Caustic (bifurcation diagram) partitioned the plane of parameters into regions, every region has either two symmetric minimum and one saddle or only one minimum. In figure (3) the number of cusp points in the plane of parameters is four and the possible numbers of critical points in every region are one of the following: 1 point (minimum), 5 points (2 symmetric minimum, 2 saddle and 1 maximum), 7 points (3 minimum, 3 saddle and 1 maximum) or 9 points (2 pairs of minimum, 2 pairs of symmetric saddle

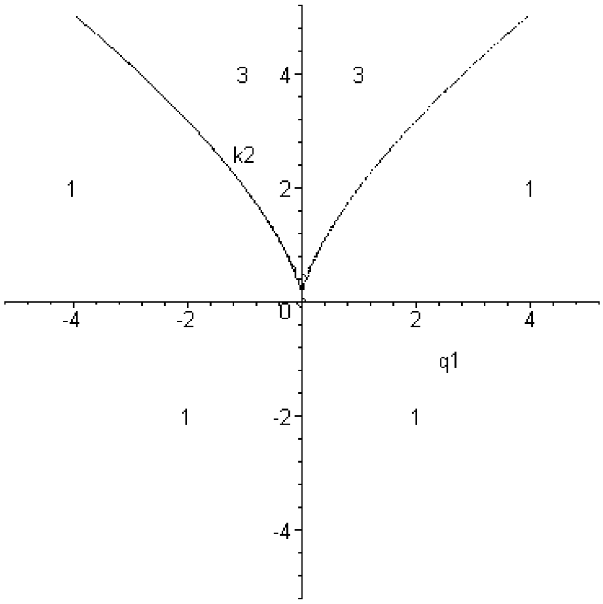

Fig. 1. Describes caustic of equation (5) when $\lambda_1 = 0$

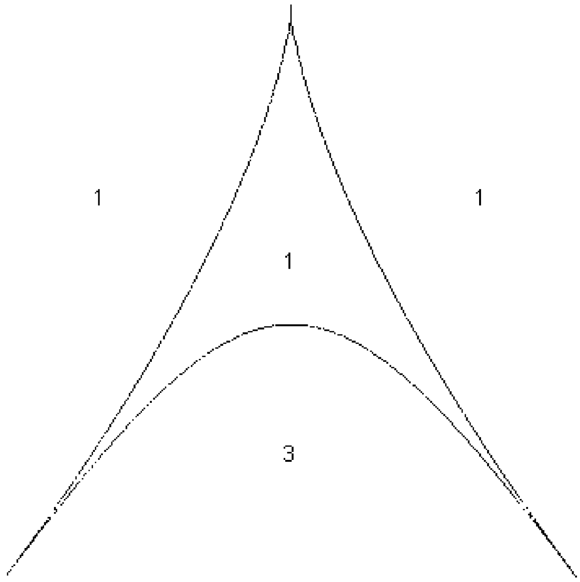

Fig. 2. Describes caustic of equation (5) when $\lambda_1 = 2$

and 1 maximum). Note that if $\lambda_2 \geq 0$, then there is only one or three solutions of system (7). For $\lambda_2 < 0$ assume that,

$$D_1 = \frac{\lambda_1^3}{27} + \frac{q_1^2}{4} \text{ and } D_2 = -\frac{(\lambda_1 + 2m)^3}{729} + \frac{q_1^2}{36},$$
$$m = -\lambda_2 > 0,$$

then system (7) have real solutions only when $\xi_1 \in (-\sqrt{m}/\sqrt{2}, \sqrt{m}/\sqrt{2})$. Thus, the possible solutions of system (7) can be found as following;

If $D_1 > 0$ and $D_2 > 0$, then system (7) may have 1 or 3 real solutions.

If $D_1 > 0$ and $D_2 < 0$, then system (7) may have 1, 3, 5 or 7 real solutions.

If $D_1 < 0$ and $D_2 > 0$, then system (7) may have 3 or 5 real solutions.

If $D_1 < 0$ and $D_2 < 0$, then system (7) may have 3, 5, 7 or 9 real solutions.

From these notations Fig. (3) have the following distributions of critical points when $\lambda_1 = 1.5$ with new 5 points (3 minimum and 2 saddle) in some domains of parameters.

The level curves of Key function $U(\xi, \delta)$ corresponding to every region are one of the following forms,

From Morse theory it is known that the union of all nondegenerates critical points of generic smooth functions can be described as a finite numbers of cells, every cell correspond to the critical point. The dimension of the cell is the

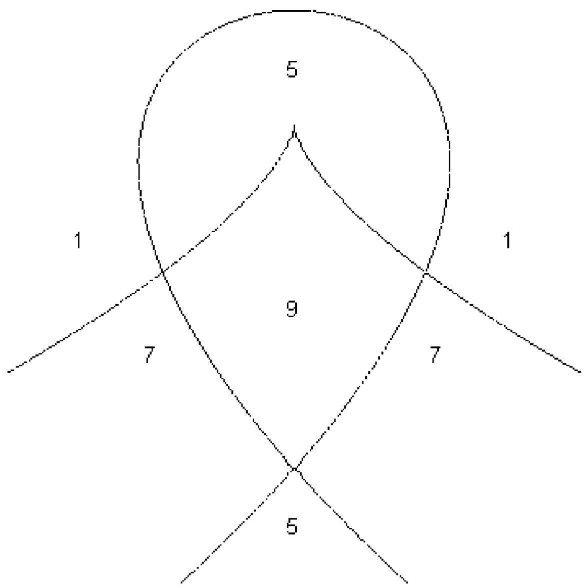

Fig. 3. Describes caustic of equation (5) when $\lambda_1 = -2$

index Morse of critical point and the mutual joining of the cells is equivalent to the mutual joining of critical points (as the singular points of Dynamical systems generated by gradient vector field). From these notations the above level curves can be described in the following graphics ,

These complexes were found by Sapronov Yu. I. [9]. Graphics in figure (6) was found by using CW-Complex, where the lines represent the saddle points, the circles represent the points of minimum and the enclosed spaces represent maximum points. Note that for small perturbation

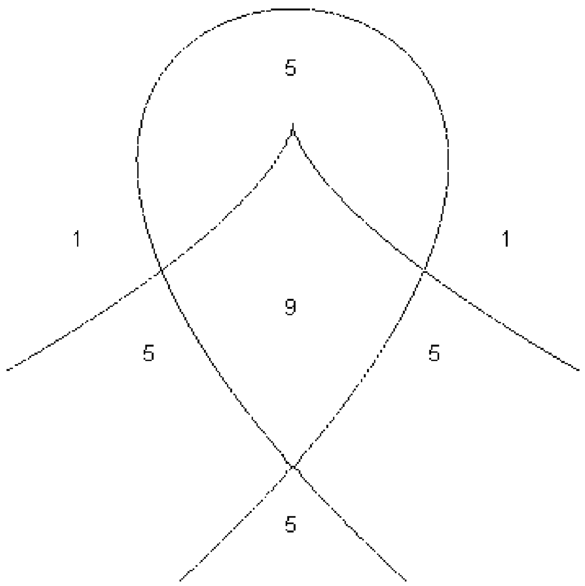

Fig. 4. Describes caustic of equation (5) when $\lambda_1 = 1.5$

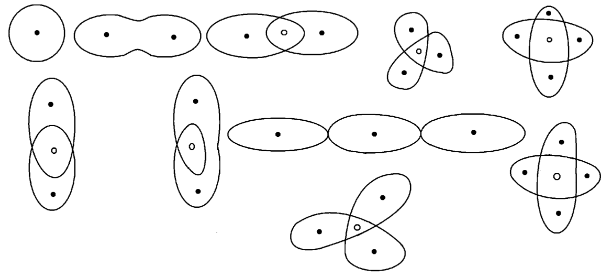

Fig. 5

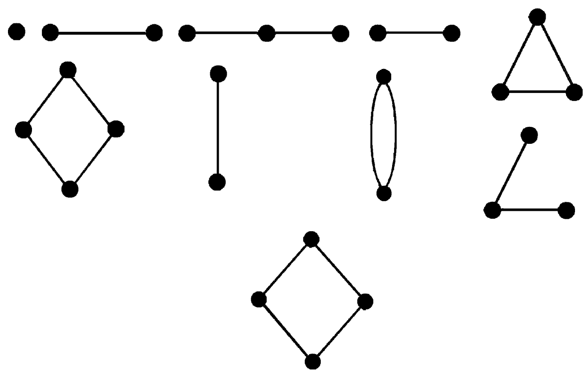

Fig. 6

of the equation $f(w, \lambda) = 0$, new branches of solutions and a new forms of caustic was found.

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