UDC 517.982

THE OPTIMAL EMBEDDING FOR THE CALDERON TYPE SPACES AND THE *J*-METHOD SPACES

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New description of the optimal target rearrangement invariant space for embedding of the Calderon spaces $\Lambda(F,E)$ in terms of the *J*-method interpolation spaces is found. We show that the corresponding Lorentz space Λ_E is involved rather than the rearrangement invariant space *E* itself.
AMS classification 46M35, 46E30, 46B70. Key words: embedding theorems, optimal spaces for embedding, interpolation spaces, real method spaces

INTRODUCTION

We consider embeddings of the Calderon spaces $\Lambda(E,F)$ to rearrangement invariant spaces. The Calderon spaces are defined with the help of the best approximation $e_t(f)_E$ of $f \in E$ by entire functions of exponential type of degree $t^{1/n}$ in each variable in the norm of rearrangement invariant space E. The space $\Lambda(F,E)$, where F is a functional lattice on $(0,\infty)$, consists of $f \in E$ such that $e_t(f)_E \in F$ with the corresponding norm or quasi-norm. These spaces are intimately connected with the Besov spaces and their generalizations. Thus embeddings $\Lambda(E,F) \subset X$ are studied along with the study of the embeddings of smooth function spaces (e.g., see [3, 5, 6]). M.Goldman and R.Kerman in [7] found the rearrangement invariant space X_0 which envelope the space $\Lambda(E,F)$. In other words they found the minimal rearrangement invariant space which contains $\Lambda(E,F)$. In the present paper we give a new description of X_0 in terms of interpolation spaces. We find that X_0 is described as a J-method interpolation space with a concrete parameter between some Lorentz space and L_∞, thus we clarify the position of this space in the family of rearrangement invariant spaces. Some conditions added to the conditions of [7] enable us to give a very transparent description of the optimal space X_0.

BASIC DEFINITIONS AND NOTATION

Everywhere below we use notation from [7]. Denote by f^* the decreasing rearrangement of a measurable function $f:\mathbb{R}^n \to \mathbb{C}$, i.e., f^* is the decreasing right continuous function on $\mathbb{R}_+$, equimeasurable to $|f(\xi)|$, i.e.,

$$mes\{\xi \in \mathbb{R}^n : |f(\xi)| > \lambda\} = \\ = mes\{x \in \mathbb{R}_+ : f^*(x) > \lambda\},$$

for all $\lambda > 0$, where *mes* denotes the Lebesgue measure on $\mathbb{R}^n$ or on $\mathbb{R}_+$ respectively.

A Banach lattice is a space E of measurable functions with a monotone norm, i.e.

$$|f| \le g, \quad g \in E \Rightarrow f \in E, \quad \|f\|_E \le \|g\|_E .$$

A Banach lattice E is called a rearrangement invariant space if

$$f^* \le g^*, g \in E \Rightarrow f \in E, \|f\|_E \le \|g\|_E .$$

We shall use axioms of the theory of Banach lattices and the theory of rearrangement invariant spaces (RIS) given by C. Bennett, R. Sharpley [1] (Chapters 1—2). In particular the Fatou property is included in the definition of Banach lattice. Thus the generalized Minkowski inequality for infinite sums and integrals is valid in RIS E.

Recall (see [1], Theorem 4.10), that there is a RIS $\widetilde{E}$ of functions determined on $\mathbb{R}_+$, such that

$$\|f\|_{E(R^n)} = \| f^* \|_{\widetilde{E}} .$$

It will cause no confusion if we use the same letter E to designate this space on $\mathbb{R}_+$.

Let $\varphi_E(t)$, where $t > 0$, be the fundamental function of RIS E, i.e.,

$$\varphi_E(t) = \left\|\chi_{(0,t)}\right\|_E .$$

It is easily seen that $\varphi_E(t)$ is quasi-concave, i.e.,

$$0 \le \varphi_E(t) \uparrow, \quad t^{-1}\varphi_E(t) \downarrow .$$

We assume without loss of generality that $\varphi_E(t)$ is concave, i.e., it is differentiable almost

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* The research was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503. The research was partially supported by Grant No. 201/05/2033 of GA ČR.

everywhere, and $0 \le \varphi_E(t) \uparrow$, $\quad 0 \le \varphi'_E(t) \downarrow$ (by using if necessary an equivalent norm in RIS E (see [1] Proposition 5.11). In what follows we assume that $\varphi_E(+0) = 0$. Without serious losses we assume also that $\varphi_E(\infty) = \infty$.

According to [7] we denote

$$\mu_E(t) = 1/\varphi_E(1/t).$$

Denote by $\mathrm{M}_{\nu,E}(\mathbb{R}^n)$ the subspace of RIS E, consisting of all entire functions of exponential type of degree $\nu > 0$ in each variable whose restrictions on $\mathbb{R}^n$ belong to E. By the Paley—Wiener theorem

$$\mathrm{M}_{\nu,E}(\mathbb{R}^n) = \{q \in E(\mathbb{R}^n) : \operatorname{supp} \hat{q} \subset (-\nu,\nu)^n\}, \quad (1)$$

where $\hat{q}$ denotes the Fourier transform of q in $\mathbb{R}^n$.

The best possible approximation of f by functions $q \in \mathrm{M}_{t^{1/n},E}(\mathbb{R}^n)$, where $t > 0$, is by definition

$$e_t(f)_E = \inf\{\|f - q\|_E : \quad q \in \mathrm{M}_{t^{1/n},E}(\mathbb{R}^n)\}. \qquad (2)$$

We omit E and t in notation in what follows when we consider the corresponding function $t \mapsto e_t(f)_E$.

Let us denote by $\widetilde{F}$ the lattice which is obtained from F by the following change of variables. We put by definition that

$$g \in \widetilde{F} \quad \text{if} \quad g(\varphi_E(1/t)) \in F,$$

or in other words

$$g \in \widetilde{F} \quad \text{if} \quad g(1/\mu_E(t)) \in F.$$

According to Sharply denote by Λ_E the Lorentz space corresponding to the concave function $\varphi_E(t)$, i.e.,

$$\Lambda_E = \left\{ f : \int_0^\infty f^*(x)\,d\varphi_E(x) < \infty \right\}$$

with the corresponding norm.

Consider the couple $\{\Lambda_E, L_\infty\}$. The key role in what follows is played by the J-method applied to the couple $\{\Lambda_E, L_\infty\}$ with the parameter space $\widetilde{F}$. Recall that by definition of the J-method, $f \in (\Lambda_E, L_\infty)^J_{\widetilde{F}}$ means that there exists a measurable vector-valued function $u(s) \in \Lambda_E \cap L_\infty$, where $0 < s < \infty$, such that

$$J(s, u(s), \{\Lambda_E, L_\infty\}) \in \widetilde{F},$$

and

$$f = \int_0^\infty u(s)\frac{ds}{s}.$$

Recall also that the J-functional is defined on the intersection of spaces of a couple $\{X_0, X_1\}$, and $J(t, u, \{X_0, X_1\}) = \max(\|u\|_{X_0}, t\|u\|_{X_1})$ for $t > 0$.

The space $(\Lambda_E, L_\infty)^J_{\widetilde{F}}$ is well defined if

$$\widetilde{F} \subset L_1(\min(1, 1/t)[dt/t], \qquad (3)$$

where $L_1(\min(1,1/t)[dt/t]$ denotes the weighted L_1 space with respect to the Haar measure on $(0,\infty)$ and the weight $\min(1,1/t)$. Thus we suppose in what follows that (3) is fulfilled.

2. DOMINATING ELEMENTS IN $(\Lambda_E, L_\infty)^J_{\widetilde{F}}$

We intend to find the simplest elements generating $(\Lambda_E, L_\infty)^J_{\widetilde{F}}$.

The following proposition is well-known for power functions $\varphi(s)$ as a particular case of the Holmstedt formula.

Proposition 2.1. *Let φ be positive concave function on $[0,\infty)$; then for any $x \in \Lambda_\varphi$ and any $t > 0$*

$$K(\varphi(t), x, \{\Lambda_\varphi, L_\infty\}) = \int_0^t x^*(s)\,d\varphi(s). \qquad (4)$$

Proof. Let $x = x_0 + x_1$, where $x_0 \in \Lambda_\varphi$, $x_1 \in L_\infty$, then

$$\int_0^t (x_0 + x_1)^*(s)\,d\varphi(s) \le \int_0^t x_0^*(s)\,d\varphi(s) + \int_0^t x_1^*(s)\,d\varphi(s) \le$$

$$\le \int_0^t x_0^*(s)\,d\varphi(s) + \varphi(t)\|x_1\|_{L_\infty}.$$

Hence

$$\int_0^t x^*(s)\,d\varphi(s) \le K(\varphi(t), x, \{\Lambda_\varphi, L_\infty\}).$$

From the other side if we consider the expansion of the form $x^*(s) = \min(x^*, \alpha) + (x^* - \min(x^*, \alpha))$, then

$$\|x^* - \min(x^*, \alpha)\|_{\Lambda_\varphi} + \varphi(t)\|\min(x^*, \alpha)\|_{L_\infty} \le$$

$$\le \|x^* - \min(x^*, \alpha)\|_{\Lambda_\varphi} + \varphi(t)\alpha$$

for any $\alpha > 0$.

Let us choose the greatest α such that the measure of the set $\{s : \min(x(s), \alpha) = \alpha\}$ is greater or equal to t. Then

$$\|x^* - \min(x^*, \alpha)\|_{\Lambda_\varphi} = \int_0^t (x^*(s) - \alpha)\,d\varphi(s).$$

Hence

$$\|x^* - \min(x^*, \alpha)\|_{\Lambda_\varphi} + \varphi(t)\alpha = \int_0^t x^*(s)\,d\varphi(s),$$

and

$$K(\varphi(t), x, \{\Lambda_\varphi, L_\infty\}) \le \int_0^t x^*(s)\,d\varphi(s).$$

Proposition is proved.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be two measure spaces, let f and g be two measurable functions on $\mathfrak{M}$ and $\mathfrak{N}$ respectively. We write

$$f \underset{\varphi}{\prec} g$$

if

$$\int_0^t f^*(x)d\varphi(x) \le \int_0^t g^*(x)d\varphi(x)$$

for all $t > 0$. We also say that f is dominated by g.

In view of Proposition 2.1 $f \underset{\varphi}{\prec} g$ is equivalent to

$$K(\varphi(t), f, \{\Lambda_\varphi, L_\infty\}(\mathfrak{M})) \le K(\varphi(t), g, \{\Lambda_\varphi, L_\infty\}(\mathfrak{N})).$$

As it is shown in [4] $f \underset{\varphi}{\prec} Cg$ is equivalent to the existence of a linear bounded operator

$$T : \{\Lambda_\varphi, L_\infty\}(\mathfrak{N}) \to \{\Lambda_\varphi, L_\infty\}(\mathfrak{M}),$$

such that $Tg = f$.

This yields that for each interpolation space X between Λ_φ and L_∞ if $f \underset{\varphi}{\prec} g$ and $g \in X$, then $f \in X$. In particular, if $f \underset{\varphi}{\prec} g$ and $g \in (\Lambda_\varphi, L_\infty)^J_{\widetilde{F}}(\mathfrak{N})$, then $f \in (\Lambda_\varphi, L_\infty)^J_{\widetilde{F}}(\mathfrak{M})$.

Evidently each $f \in (\Lambda_\varphi, L_\infty)^J_{\widetilde{F}}(\mathfrak{M})$ is dominated by $f^* \in (\Lambda_\varphi, L_\infty)^J_{\widetilde{F}}(0,\infty)$. We intend to show that actually each $f \in (\Lambda_\varphi, L_\infty)^J_{\widetilde{F}}(\mathfrak{M})$ is dominated by a rather simple $g \in (\Lambda_\varphi, L_\infty)^J_{\widetilde{F}}(0,\infty)$.

Recall that practically we are interested in studying of Λ_{φ_E}. From now on we consider $\Lambda_E = \Lambda_{\varphi_E}$ only, and denote for brevity $f \underset{\varphi_E}{\prec} g$ by $f \underset{E}{\prec} g$.

Let $f \in (\Lambda_E, L_\infty)^J_{\widetilde{F}}(\mathfrak{M})$. By definition

$$f = \int_0^\infty u(s)ds/s$$

for some measurable vector-valued function $u(s)$ on $(0,\infty)$ with values in $L_\infty \cap \Lambda_E$ such that $J(s, u(s), \{\Lambda_E, L_\infty\}) \in \widetilde{F}$. Without loss of generality we assume that $u(s) \ge 0$.

We intend to transform f. First we increase $u(s)$ such that $\| u(s) \|_{\Lambda_E} = s \| u(s) \|_{L_\infty}$ for all $s > 0$, while the corresponding f remains in $(\Lambda_E, L_\infty)^J_{\widetilde{F}}$. Then we change $u(s)$ by $u(s)^*$. The corresponding integral is a function in $(\Lambda_E, L_\infty)^J_{\widetilde{F}}(0,\infty)$, and we denote it by $f_*(x)$, where $x \in (0,\infty)$. We conclude $f \underset{E}{\prec} f_*$.

Indeed,

$$K(\varphi_E(t), f, \{\Lambda_E, L_\infty\}) \le$$

$$\le \int_0^\infty K(\varphi_E(t), u(s), \{\Lambda_E, L_\infty\})ds/s =$$

$$= \int_0^\infty (\int_0^t u(s)^*(x)d\varphi_E(x))ds/s =$$

$$= \int_0^t (\int_0^\infty u(s)^* ds/s)(x)d\varphi_E(x) =$$

$$= K(\varphi_E(t), f_*, \{\Lambda_E, L_\infty\}).$$

And finally we change $u(s)^*$ by $\chi_{(0,\alpha(s))}h(s)/s$, where $h(s) = J(s, u(s), \{\Lambda_E, L_\infty\})$ and $\alpha(s) = \varphi_E^{-1}(s)$. Thus $h \in \widetilde{F}$. Denote

$$f_{**} = \int_0^\infty \chi_{(0,\alpha(s))}h(s)/s \frac{ds}{s}.$$

Again the K-functional of f_{**} is greater than the K-functional of f_*, since for any $u(s) \in \Lambda_E \cap L_\infty$

$$K(\varphi_E(t), u(s), \{\Lambda_E, L_\infty\}) \le$$

$$\le \min(\| u(s) \|_{\Lambda_E}, \varphi_E(t) \| u(s) \|_{L_\infty}) =$$

$$= \min(\| u(s) \|_{\Lambda_E}, \frac{\varphi_E(t)}{s} \| u(s) \|_{\Lambda_E}) =$$

$$= \min\left(1, \frac{\varphi_E(t)}{s}\right) \| u(s) \|_{\Lambda_E} =$$

$$= K(\varphi_E(t), \chi_{(0,\alpha(s))}, \{\Lambda_E, L_\infty\}) \| u(s) \|_{\Lambda_E} \ /s.$$

Hence

$$K(\varphi_E(t), f_*, \{\Lambda_E, L_\infty\}) =$$

$$= \int_0^\infty K(\varphi_E(t), u(s)^*, \{\Lambda_E, L_\infty\})ds / s \le$$

$$\le \int_0^\infty K(\varphi_E(t), \chi_{(0,\alpha(s))}h(s) / s, \{\Lambda_E, L_\infty\})ds / s =$$

$$= K(\varphi_E(t), f_{**}, \{\Lambda_E, L_\infty\})$$

in view of integral form of the K-functional.

Thus we see $f \underset{E}{\prec} f_{**}$. At the same time

$$\left\|\chi_{(0,\alpha(s))}h(s)/s\right\|_{L_\infty} = h(s)/s,$$

$$\left\|\chi_{(0,\alpha(s))}h(s)/s\right\|_{\Lambda_E} = \varphi_E(\alpha(s))h(s)/s = h(s).$$

Hence

$$J(s, \chi_{(0,\alpha(s))}h(s)/s, \{\Lambda_E, L_\infty\}) = h(s),$$

which means $f_{**} \in (\Lambda_E, L_\infty)^J_{\widetilde{F}}$.

These elements f_{**} and the corresponding integrals are intimately connected with the Hardy operator, considered in [7]. Indeed,

$$f_{**}(t) = \int_0^\infty \chi_{(0,\alpha(s))}(t) h(s)/s \frac{ds}{s} = \int_{\alpha^{-1}(t)}^\infty h(s)\frac{ds}{s^2}. \quad (5)$$

If we naturally transform (5) we obtain

$$f_{**}(t) = \int_{\alpha^{-1}(t)}^\infty \frac{h(s)}{s^2} ds = \int_0^{1/\alpha^{-1}(t)} h(1/s)ds =$$

$$= \int_0^{1/\varphi_E(t)} h(1/s)ds = \int_0^{\mu_E(1/t)} h(1/s)ds =$$

$$= \int_0^{1/t} h\left(\frac{1}{\mu_E(\tau)}\right) d\mu_E(\tau) = \int_0^{1/t} h(\varphi_E(1/\tau)) d\mu_E(\tau) =$$

$$= \int_0^{1/t} g(\tau) d\mu_E(\tau),$$

where $g \in F$, since $h \in \widetilde{F}$.

Thus we obtain that for each function $g \in F(0,\infty)$ the integral

$$\int_0^{1/t} g(\tau) d\mu_E(\tau) \quad (6)$$

is a function from $(\Lambda_E, L_\infty)^J_{\widetilde{F}}(0,\infty)$. Following to [7] we introduce the operator

$$H : g \mapsto \int_0^{1/t} g(\tau) d\mu_E(\tau),$$

which is called the Hardy operator. Thus we see that

$$H : F(0,\infty) \to (\Lambda_E, L_\infty)^J_{\widetilde{F}}(0,\infty).$$

3. EMBEDDING TO THE J-METHOD SPACES

It is more convenient now to consider the Hardy operator (6) in the form

$$\int_0^{1/t} g(\tau) d\mu_E(\tau) = \int_t^\infty g(1/\tau) \left| d\frac{1}{\varphi_E(\tau)} \right|,$$

where $t > 0$.

Recall that

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(x)dx.$$

Proposition 3.1. *Let $E = E(\mathbb{R}^n)$ be a RIS and $T > 0$. There is a constant c independent of T, such that for all $f \in E$ and $t \in (0,T)$*

$$f^{**}(t) \le c\left(\frac{1}{\varphi_E(T)} \| f \|_E + H(e(f))(t)\right), \quad (7)$$

where $e(f)$ is defined by (2).

Proof. Fix $t \in (0,T)$. Let $N \in \mathbb{N}$ be such that

$$2^N \varphi_E(t) \le \varphi_E(T) < 2^{N+1}\varphi_E(t).$$

Define the sequence $t_0, t_1, t_2, \dots, t_N$ by

$$\varphi_E(t_i) = 2^i \varphi_E(t). \quad (8)$$

For each $i = 0,1,2,\dots,N-1$ there is a decomposition $f = b_i + g_i$, where $g_i \in M_{t_i^{-1/n},E}$ such that

$$\|b_i\|_E \le 2e_{\frac{1}{t_i}}(f)_E. \quad (9)$$

Define $a_i \in M_{t_i^{-1/n},E}$ by

$$a_i = b_{i+1} - b_i = g_i - g_{i+1}, \ (i = 0,1,\dots,N-2). \quad (10)$$

Then $f = b_0 + \sum_{i=0}^{N-2} a_i + g_{N-1}$ and

$$f^{**}(t) \le b_0^{**}(t) + \sum_{i=0}^{N-2} \|a_i\|_\infty + \|g_{N-1}\|_\infty. \quad (11)$$

Using the well-known inequality of different metrics for entire functions of exponential type (see [2], Theorem 1), together with (10), (9) and (8), we obtain

$$\|a_i\|_\infty \le \frac{c}{\varphi_E(t_i)}\|a_i\|_E \le \frac{c}{\varphi_E(t_i)}\left(\|b_{i+1}\|_E + \|b_i\|_E\right) \le$$

$$\le \frac{2c}{\varphi_E(t_i)}\left(e_{\frac{1}{t_{i+1}}}(f)_E + e_{\frac{1}{t_i}}(f)_E\right) \le \frac{2c}{\varphi_E(t_i)} e_{\frac{1}{t_{i+1}}}(f)_E \le \quad (12)$$

$$\le 16c\int_{t_{i+1}}^{t_{i+2}} e_{\frac{1}{\tau}}(f)_E \left| d\frac{1}{\varphi_E(\tau)} \right|.$$

By same way we deduce

$$\|g_{N-1}\|_\infty \le \frac{c}{\varphi_E(t_{N-1})}\|g_{N-1}\|_E \le$$

$$\le \frac{c}{\varphi_E(t_{N-1})}(\|f\|_E + \|b_{N-1}\|_E) \le$$

$$\le \frac{4c}{\varphi_E(T)}\|f\|_E + \frac{2c}{\varphi_E(t_{N-1})} e_{\frac{1}{t_{N-1}}}(f)_E \le \quad (13)$$

$$\le \frac{4c}{\varphi_E(T)}\|f\|_E + 2c\int_{t_{N-1}}^{t_N} e_{\frac{1}{\tau}}(f)_E \left| d\frac{1}{\varphi_E(\tau)} \right|.$$

On the other hand, an application of the well-known estimate for rearrangements (see [1], Proposition 5.9) gives

$$b_0^{**}(t) \le \frac{1}{\varphi_E(t)}\|b_0\|_E \le \frac{2}{\varphi_E(t)} e_{\frac{1}{t}}(f)_E \le$$

$$\le 4\int_t^{t_1} e_{\frac{1}{\tau}}(f)_E \left| d\frac{1}{\varphi_E(\tau)} \right|.$$

Substituting this estimate, (12) and (13) into (11), we obtain the desired conclusion (7).

Proposition is proved.

Corollary 3.1. *If* $\varphi_E(\infty)=\infty$ *then*

$$f^{**}\le cH(e(f)). \qquad (14)$$

Proof. Take a limit in (7) as $T\to\infty$.

Corollary is proved.

Corollary 3.2. *The Calderon type space* $\Lambda(E,F)$ *is contained in* $(\Lambda_E,L_\infty)^J_{\widetilde F}$.

Proof. Since $f^*\le f^{**}$, we conclude

$$f^*\le cH(e(f))$$

on $(0,\infty)$ by (14). If $e(f)\in F$, then $H(e(f))\in\in(\Lambda_E,L_\infty)^J_{\widetilde F}$, and thereby $f\in(\Lambda_E,L_\infty)^J_{\widetilde F}$.

Corollary is proved.

By definition $\Lambda(E,F)\subset E$, hence

$$\Lambda(E,F)\subset(\Lambda_E,L_\infty)^J_{\widetilde F}\cap E. \qquad (15)$$

As we see below $(\Lambda_E,L_\infty)^J_{\widetilde F}\cap E$ sometimes is the smallest RIS which contains $\Lambda(E,F)$.

4. OPTIMALITY OF THE SPACE $(\Lambda_E,L_\infty)^J_{\widetilde F}$

The above mentioned optimality is based on the results of [7]. First we study the condition which was used in the paper [7].

The following Proposition is a consequence of the change of variables and definition of the space $\widetilde F$.

Proposition 4.1. *The operator*

$$G[g](t)=\int_t^\infty g(\tau)\frac{d\mu_E(\tau)}{\mu_E(\tau)}$$

is bounded from F to F if and only if the operator

$$\widetilde G[f](t)=\int_0^t f(s)\frac{ds}{s}$$

is bounded from $\widetilde F$ to $\widetilde F$.

Theorem 4.1. *If* $G:F\to F$, *then the smallest RIS* X_0, *which contains* $\Lambda(E,F)$ *coincides with* $(\Lambda_E,L_\infty)^J_{\widetilde F}$ *on any finite interval* $(0,T_0)$.

Proof. In view of Corollary 3.2 it is necessary to prove the inclusion $(\Lambda_E,L_\infty)^J_{\widetilde F}\subset X_0$, since the reverse inclusion is already proved.

The existence of the optimal space X_0 was proved in [7] under condition $G:F\to F$, and it was shown that $\|f\|_{X_0}$ is equivalent to

$$\sup_{\varphi\in\Omega'_{EF}}\int_0^\infty-\varphi'(t)f_E(t)dt+\left\|f^*\chi_{(T,\infty)}\right\|_E,$$

for arbitrary T, where

$$f_E(t)=\mu_E(t)\int_0^{\beta(t)}f^*(\tau)d\varphi_E(\tau)=$$
$$=\mu_E(t)K(\varphi_E(\beta(t)),f,\{\Lambda_E,L_\infty\}).$$

The nature of the set Ω'_{EF} and the function β is of no importance in what follows.

We see that the norm is equivalent to

$$\sup_{\varphi\in\Omega'_{EF}}\int_0^\infty-\varphi'(t)f_E(t)dt$$

on the interval $(0,T_0)$. Thus we see that the norm in X_0 on the interval $(0,T_0)$ is K-monotone with respect to the couple $\{\Lambda_E,L_\infty\}$. Hence X_0 is an interpolation space between Λ_E and L_∞ on the interval $(0,T_0)$.

Let f be an arbitrary element of $(\Lambda_E,L_\infty)^J_{\widetilde F}$ on $(0,T_0)$ and f_{**} be the corresponding dominating element. If we find $y\in\Lambda(E,F)$ such that $f\underset{E}{\prec}f_{**}\underset{E}{\prec}y$, then $f\in X_0$ on $(0,T_0)$, and Theorem will be proved.

Indeed, by definition $y\in X_0$. The interpolation property of the space X_0 between Λ_E and L_∞ on $(0,T_0)$ and $f\underset{E}{\prec}y$ implies that $f\in X_0$. Hence $(\Lambda_E,L_\infty)^J_{\widetilde F}\subset X_0$ on $(0,T_0)$.

Thus we return to

$$f_{**}=\int_0^T u(s)\frac{ds}{s},$$

where $u(s)=\chi_{(0,\alpha(s))}h(s)/s$ and $h\in\widetilde F$. We can take finite T because $L_\infty\subset\Lambda_E$ on $(0,T_0)$.

Recall (e.g., see [2]) that for $x\in R$

$$\chi_{(-1/\nu,1/\nu)}(x)\le C\nu^{-2}\frac{\sin^2(2^{-1}\nu x)}{x^2},$$

where C is a universal constant. Hence for $\xi\in R^n$

$$\prod_{j=1}^n\chi_{(-1/t^{1/n},1/t^{1/n})}(\xi_j)\le C^nt^{-2}\prod_{j=1}^n\frac{\sin^2(2^{-1}t^{1/n}\xi_j)}{\xi_j^2}=q_t(\xi),$$

where $q_t(\xi)$ is an entire function of exponential type $t>0$.

Thereby

$$\chi_{(0,1/t)}\le q_t^*. \qquad (16)$$

Let

$$y=\int_0^T q_{1/\alpha(s)}h(s)/s\frac{ds}{s}.$$

Because of (16) we evidently have $f_{**}\underset{E}{\prec}y$. Let us show that $y\in\Lambda(E,F)$.

Recall that

$$\left\|\chi_{(0,1/t)}\right\|_E\asymp\|q_t\|_E$$

(see [2]).

First we get $y\in E$, since

$$\|y\|_E \le \int_0^T \|q_{1/\alpha(s)}\|_E\, h(s)/s \frac{ds}{s} \le$$

$$\le C\int_0^T \|\chi_{(0,\alpha(s))}\|_E\, h(s)/s \frac{ds}{s} = C\int_0^T h(s)\frac{ds}{s} < \infty$$

in view of $h \in \widetilde{F} \subset L_1(\min(1,1/t)[dt/t]$ (see (3)).

It is easily seen that for sufficiently large $t > 0$

$$y_t = \int_{1/\mu_E(t)}^T q_{1/\alpha(s)} h(s)/s \frac{ds}{s} \in M_{t^{-1/n},E},$$

since for $s > 1/\mu_E(t)$, which is equivalent to $t > 1/\alpha(s)$, we have $q_{1/\alpha(s)} \in M_{t^{-1/n},E}$.

Furthermore

$$e_t(y) \le \|y - y_t\|_E = \left\| \int_0^{1/\mu_E(t)} q_{1/\alpha(s)} h(s)/s \frac{ds}{s} \right\|_E \le$$

$$\le \int_0^{1/\mu_E(t)} \|q_{1/\alpha(s)}\|_E\, h(s)/s \frac{ds}{s} \le \int_0^{1/\mu_E(t)} h(s)\frac{ds}{s} =$$

$$= \int_t^\infty h(1/\mu_E(\tau))\frac{d\mu_E(\tau)}{\mu_E(\tau)} = G(g)(t),$$

where $g(t) = h(1/\mu_E(t))$.

Hence $e_t(y) \in F$ since $h(1/\mu_E(t) \in F$. Thus $y \in \Lambda(E,F)$.

Theorem is proved.

Corollary 4.1. *If $G : F \to F$, then the smallest RIS containing $\Lambda(E,F)$ is equal to $(\Lambda_E, L_\infty)^J_{\widetilde{F}} \cap E$.*

Proof. In view (15) we have to prove $(\Lambda_E, L_\infty)^J_{\widetilde{F}} \cap E \subset X_0$. Recall (e.g., see [2]) that the smallest RIS containing $M_{t,E}$ is $E \cap L_\infty$. This yieldsthat $E \cap L_\infty \subset X_0$. Thusif $f \in (\Lambda_E, L_\infty)^J_{\widetilde{F}} \cap E$, then $f^*\chi_{(0,T)} \in X_0$ by Theorem 4.1 and $f^*\chi_{(T,\infty)} \in E \cap L_\infty \subset X_0$. Hence $f^* \in X_0$ and we conclude $(\Lambda_E, L_\infty)^J_{\widetilde{F}} \cap E \subset X_0$.

Corollary is proved.

Now we try to find an explicit description of the optimal space X_0.

The following Proposition is similar to Proposition 4.1 and we also leave the proof to the reader, since it may be obtained by direct change of variables.

Proposition 4.2. *The operator*

$$G_0[g](t) = \frac{1}{\mu_E(t)}\int_0^t g(\tau)d\mu_E(\tau)$$

is bounded from F to F if and only if the operator

$$\widetilde{G_0}[f](t) = t\int_t^\infty f(s)\frac{ds}{s^2}$$

is bounded in $\widetilde{F}$.

The sum of the operators $\widetilde{G}$ and $\widetilde{G_0}$ is equal to the Calderon operator

$$S[f](t) = \int_0^\infty f(s)\min(1,\frac{t}{s})\frac{ds}{s} = \int_0^t f(s)\frac{ds}{s} + t\int_t^\infty f(s)\frac{ds}{s^2}.$$

Corollary 4.2. *The Calderon operator*

$$S[f](t) = \int_0^\infty f(s)\min(1,\frac{t}{s})\frac{ds}{s}$$

is bounded from $\widetilde{F}$ to $\widetilde{F}$, if and only if G and G_0 are bounded in F.

Recall that if the Calderon operator S maps the parameter space $\widetilde{F}$ into itself, then the J-method space $(X_0,X_1)^J_{\widetilde{F}}$ coincides with the K-method space with the same parameter space (e.g. [1]).

Hence

$$(\Lambda_E, L_\infty)^J_{\widetilde{F}} = (\Lambda_E, L_\infty)^K_{\widetilde{F}}, \tag{17}$$

which means that

$$(\Lambda_E, L_\infty)^J_{\widetilde{F}} = \{f : K(t,f,\{\Lambda_E, L_\infty\}) \in \widetilde{F}\}.$$

This formula gives us opportunity to calculate the space $(\Lambda_E, L_\infty)^J_{\widetilde{F}}$ in terms of the K-functional of the couple $\{\Lambda_E, L_\infty\}$.

If we combine (17) and (4), we conclude that $f \in (\Lambda_E, L_\infty)^J_{\widetilde{F}}(0,\infty)$,

$$K(t,f,\{\Lambda_E, L_\infty\}) = \int_0^{\varphi_E^{-1}(t)} f^*(s)d\varphi_E(s) \in \widetilde{F},$$

and

$$\int_0^{1/t} f^*(s)d\varphi_E(s) \in F.$$

are equivalent.

We intend to apply these considerations to the restriction of spaces Λ_E and L_∞ onto a finite interval $(0,T_0)$. In this case identity (17) takes place if the Calderon operator S is bounded in the space $\widetilde{F}(0,T)$ for some finite T. The operator S is bounded in $\widetilde{F}(0,T)$ if G and G_0 are bounded in $F(T_1,\infty)$ for some $0 < T_1 < \infty$.

Thus we obtain a new description of the optimal target space for embedding of the Calderon type spaces.

Theorem 4.2. *If for some T_1 the space $F(T_1,\infty)$ is invariant under the operators*

$$G[g](t) = \int_t^\infty g(\tau)\frac{d\mu_E(\tau)}{\mu_E(\tau)}$$

and

$$G_0[g](t) = \frac{1}{\mu_E(t)}\int_0^t g(\tau)d\mu_E(\tau),$$

then the optimal RIS X_0 for the embedding $\Lambda(E,F) \subset X$ consists of f such that

$$\int_0^{1/t} f^*(s)\chi_{(0,T)}(s)d\varphi_E(s) \in F \text{ and } f^*\chi_{(T,\infty)} \in E$$

for some $T > 0$.

V. I. Ovchinnikov expresses his gratitude to the Mathematical Institute of the Czech Academy of Science for its kind hospitality during the preparation of this work.

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