МАТЕМАТИКА

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ON A SECOND ORDER DIFFERENTIAL INCLUSION WITH RANDOM PERTURBATION OF VELOCITY*

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A second order stochastic differential inclusion in R^n is investigated that describes the motion of a mechanical system with set-valued deterministic lower semicontinuous force and random influence on the velocity. The notion of relaxed solution is introduced and an existence theorem for that is proved.

It is a well-known fact that a second order differential equation $\ddot{x}(t) = F(t, x(t), \dot{x}(t))$ is represented as a first order system on the space of doubled dimension

$$\begin{cases} \dot{x}(t) = v(t) \\ \dot{v}(t) = F(t, x(t), v(t)). \end{cases}$$

For the sake of simplicity let us call the first equation of above system horizontal and the second one vertical. As it was mentioned in [4], this leads to several possibilities for arising random perturbations in the second order equation: when the randomness appears in the vertical equation, in the horizontal one and in both equations. All three cases have natural physical meaning and the corresponding stochastic equations describe three kinds of mechanical systems with randomness.

A more interesting and complicated problem is to consider the situation when the right-hand side of the second order equation is set-valued, i.e., the equations is in fact replaced by a second order differential inclusion. Such inclusions with randomness describe mechanical systems with discontinuous forces or with control under the influence of stochastic parameters. It should be pointed out that the types of stochastic differential inclusions, corresponding to three types of stochastic perturbations, mentioned above, require different methods for their investigation.

In [5] we studied the case where the vertical equation was perturbed by a stochastic term expressed via white noise. The physical meaning of this case is that there is a stochastic summand in the force field of mechanical system. Inclusions of this sort are called the stochastic differential inclusions of Langevin type since for single-valued forces they are transformed into the well-known Langevin's equations. In [5] the general case of Langevin type inclusions on Riemannian manifolds was considered that allowed us to cover the mechanical systems on non-linear configuration spaces.

In this paper we present the first attempt of investigating a second order differential inclusion with random perturbation of the horizontal part. We consider the systems in Euclidean space R^n whose set-valued force fields are lower semicontinuous.

Preliminary information from the theory of set-valued maps and differential inclusions can be found, e.g., in [1], [6].

Let $F(t,x)$ be a lower semi-continuous set-valued map $F : R \times R^n \to 2^{R^n}$ with closed images and $A(t,x) : R^n \to R^n$ be a field of single-valued linear operators jointly continuous in parameters $t \in R$ and $x \in R^n$. We suppose that $F(t,x)$ and $A(t,x)$ satisfy the so called Itô condition, i.e., that there exists a constant $\Theta > 0$ such that

$$\|F(t,x)\| + \|A(t,x)\| < \Theta(1 + \|x\|) \tag{1}$$

for all $t \in R$ and $x \in R^n$ where $\|A(t,x)\|$ is the operator norm and $\|F(t,x)\| = \sup_{y \in F(t,x)} \|y\|$.

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Consider the following system

$$\begin{cases}\dot{x}(t) = a(t) + A(t, x(t))\dot{w}(t) \\ \dot{a}(t) \in F(t, x(t)),\end{cases} \tag{2}$$

where $\dot{w}(t)$ is Itô white noise.

As usual, (2) will take exact mathematical sense after transition to integral form. In what follows we consider R^n and R with their Borel σ-algebras $\mathcal{B}^n$ and $\mathcal{B}$, respectively. Let $x(\cdot)$ be a continuous curve. Consider the set-valued vector field $F(t, x(t))$ along $x(\cdot)$ and denote by $\mathcal{P}F(\cdot, x(\cdot))$ the set of all measurable selections of $F(t, x(t))$, i.e., the set of measurable maps $\{f : R \to R^n : f(x(t)) \in F(t, x(t))\}$. It is obvious that since condition (1) is satisfied, all those selections are integrable on any finite interval in R with respect to Lebesgue measure. Denote by $\int \mathcal{P}F(\cdot, x(\cdot))$ the set of integrals with varying upper limits of those selections. Now we can transform (2) into

$$\begin{cases}x(t) = x_0 + \int_0^t a(s)ds + \int_0^t A(s, x(s))dw(s) \\ a(\cdot) \in a_0 + \int \mathcal{P}F(\cdot, x(\cdot))\end{cases} \tag{3}$$

where $\int_0^t A(t, x(t))dw(t)$ is Itô integral and $a(t)$ is a continuous (random) vector field along $x(t)$, i.e., $a_\omega(t)$ is a continuous vector field along a sample trajectory $x_\omega(t)$ of $x(t)$.

One can easily see that such a system corresponds to a second order differential inclusion of above-mentioned sort. Obviously (2) describes the mechanical systems with deterministic set-valued forces, subjected to a random influence on velocities expressed in terms of white noise. In another language of (3) this means that the (random) trajectory of the system is described by an Itô diffusion-type process under the action of set-valued deterministic force.

Recall some facts and notions involved in further considerations. Specify $l > 0$. In what follows we denote by λ the normalized Lebesgue measure on $[0, l]$, i.e., such that $\lambda([0, l]) = 1$.

Definition 1 *Let $\mathcal{E}$ be a separable Banach space. A non-empty set $\mathcal{M} \subset L^1([0, l]; \mathcal{E})$ is called decomposable, if for any $f, g \in \mathcal{M}$ and any measurable subset M in $[0, l]$,*

$$f \circ \kappa_M + g \circ \kappa_{[0,l]\setminus M} \in \mathcal{M},$$

where κ_N is the characteristic function of the set N (see [2] and [6] for details).

Lemma 2. *Let (Ξ, d) be a separable metric space, X be a Banach space. Consider the space $Y = L^1(([0, l], \mathcal{B}, \lambda), X))$ of integrable maps from $[0, l]$ into X. If a set-valued map $G : \Xi \to Y$ is lower semicontinuous and has closed decomposable images, it has a continuous selection.*

This is a particular case of the well-known Bressan–Colombo Theorem (see, e.g., Lemma 9.2 in [2]).

Definition 3 *We say that (3) has a relaxed solution on $[0, l] \subset R$ with initial position $x_0 \in R^n$ and initial velocity $a_0 \in R^n$, if there exists a probability space $(\Omega, \mathcal{F}, P)$ and three stochastic processes in R^n, given on $(\Omega, \mathcal{F}, P)$: a stochastic process $x(t)$, $x(0) = x_0$, with continuous sample paths, a Wiener process $w(t)$, adapted to $x(t)$, and a stochastic process $a(t)$, $a(0) = a_0$, with continuous sample paths, adapted to $x(t)$, such that*

$$x(t) = x_0 + \int_0^t a(s)ds + \int_0^t A(s, x(s))dw(s)$$

satisfied for almost all $t \in [0, l]$ P-a.s. and at any $t \in [0, l]$ P-a.s. $a(t) = a_0 + \int_0^t f^{(t)}(x(\tau))d\tau$ for a certain selection $f^{(t)}(x(\tau))$ from $\mathcal{P}F(\cdot, x(\cdot))$ continuously depending on t with respect to the topology of $L^1(([0, l], \mathcal{B}, \lambda), R^n)$.

Denote by $C^0([0, l], R^n)$ the Banach space of continuous maps from $[0, l]$ to R^n (i.e., continuous curves in R^n, given on $[0, l]$).

Theorem 4. *Let, as mentioned above, $F(t,x)$ be a lower semi-continuous set-valued map $F : R \times R^n \to 2^{R^n}$ with closed images and $A(t, x) : R^n \to R^n$ be a field of single-valued linear operators jointly continuous in parameters $t \in R$ and $x \in R^n$. Let also (1) be fulfilled. Then for any specified $l > 0$, $x_0, a_0 \in R^n$ inclusion (3) has a relaxed solution on $[0, l]$ with initial position x_0 and initial velocity a_0.*

Proof. In $C^0([0, l], R^n)$ introduce the σ-algebra $\tilde{\mathcal{F}}$ generated by cylindrical sets. By $\tilde{\mathcal{F}}_t$ denote the σ-algebra generated by cylindrical sets over $[0, t] \subset [0, l]$.

Consider the set-valued map B sending $x(\cdot) \in C^0([0, l], R^n)$ into $\mathcal{P}F(\cdot, x(\cdot))$. Since under condition (1) all selections from $\mathcal{P}F(\cdot, x(\cdot))$ are integrable (see above), B takes values in the space $L^1(([0, l], \mathcal{B}, \lambda), R^n)$. It is known (see, e.g., § 5.5 from [6]) that under the above-mentioned conditions $B : C^0([0, l], R^n) \to L^1(([0, l], \mathcal{B}, \lambda), R^n)$ is lower semicontinuous and for any $x(\cdot) \in C^0([0, l], R^n)$ the set $\mathcal{P}F(\cdot, x(\cdot))$, i.e., the image $B(x(\cdot))$ is decomposable and closed. Thus, by Lemma 2 B has a continuous selection $b : C^0([0, l], R^n) \to L^1(([0, l], \mathcal{B}, \lambda), R^n)$.

For any $t \in [0, l]$ introduce the map $f_t : C^0([0, l], R^n) \to C^0([0, l], R^n)$ that sends a curve $x(\cdot) \in C^0([0, l], R^n)$ into the curve

$$f_t(\tau, x(\cdot)) = \begin{cases} x(\tau) \text{ for } \tau \in [0, t] \\ x(t) \text{ for } \tau \in [t, l] \end{cases}.$$

Obviously the map f_t is continuous. Since $f_t(\tau, x(\cdot))$ belongs to $C^0([0, l], R^n)$, the curve $b(f_t(\tau, x(\cdot))) \in L^1(([0, l], \mathcal{B}, \lambda), R^n)$ is well-posed. By the construction $b(f_t(\tau, x(\cdot))) \in F(\tau, x(\tau))$ for almost all $\tau \in [0, t]$ and this selection continuously depends on t in $L^1(([0, l], \mathcal{B}, \lambda), R^n)$.

Consider the map $a : [0, l] \times C^0([0, l], R^n) \to R^n$ defined by the formula

$$a(t, x(\cdot)) = a_0 + \int_0^t b(f_t(\tau, x(\cdot)))d\tau. \qquad (4)$$

By the construction this map is continuous jointly in $t \in [0, l]$ and $x(\cdot) \in C^0([0, l], R^n)$. In addition it is obvious that if $x_1(\cdot)$ and $x_2(\cdot)$ coincide on $[0, t]$ then $a(t, x_1(\cdot)) = a(t, x_2(\cdot))$. This means that $a(t, x(\cdot))$ is measurable with respect to $\tilde{\mathcal{F}}_t$. (see, e.g., [3]).

Taking into account (1) one can easily derive the inequality

$$\|a(t, x(\cdot))\| = \left\|\int_0^t b(f_t(\tau, x(\cdot)))d\tau\right\| \le \int_0^t \|b(f_t(\tau, x(\cdot)))\| d\tau \le$$
$$\le \int_0^t \|F(\tau, x(\tau))\| d\tau \le \Theta \int_0^t (1 + \|x(\tau)\|)d\tau \le$$
$$\le \Theta \int_0^t (1 + \|x(\cdot)\|_{C^0})ds \le l\Theta(1 + \|x(\cdot)\|_{C^0})$$

where $\|\cdot\|_{C^0}$ is the norm in $C^0([0, l], R^n)$.

Introduce $A(t, x(\cdot))$ as $A(t, x(\cdot)) = A(t, x(t))$. Notice that $A(t, x(\cdot))$ is measurable with respect to $\tilde{\mathcal{F}}_t$ and that from (1) it follows that $\|A(t, x(\cdot))\| \le \Theta(1 + \|x(\cdot)\|_{C^0})$. So, both $a(t, x(\cdot))$ and $A(t, x(\cdot))$ satisfy the Itô condition in the form

$$\|a(t, x(\cdot)\| + \|A(t, x(\cdot)\| \le \bar{\Theta}(1 + \|x(\cdot)\|_{C^0})$$

with $\bar{\Theta} = max(\Theta, l\Theta)$.

Now the couple $a(t, x(\cdot))$ and $A(t, x(\cdot))$ satisfies all conditions of theorem III.2.4 from [3], hence, the stochastic differential equation

$$x(t) = x_0 + \int_0^t a(s, x(\cdot))ds + \int_0^t A(s, x(\cdot))dw(s)$$

has a weak solution on $[0, l]$. This means that there exist a probabilistic measure μ on $(C^0([0, l], R^n), \tilde{\mathcal{F}})$ and a Wiener process in R^n, given on the probability space $(C^0([0, l], R^n), \tilde{\mathcal{F}}, \mu)$ and adapted to $\tilde{\mathcal{F}}_t$, such that the coordinate process $x(t)$ on $(C^0([0, l], R^n), \tilde{\mathcal{F}}, \mu)$ and $w(t)$ satisfy (5). This together with (4) completes the proof of Theorem. □

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