ON SOME QUESTIONS OF THE SPECTRAL THEORY OF LINEAR RELATIONS

© 2002 г. A. G. Baskakov, K. I. Chernyshov*

Voronezh State University
**Voronezh State Forestry Academy*

§ 1. Introduction

The present paper[1] is devoted to the investigation of certain questions of the spectral theory of linear relations (multivalued linear operators) as well as to the construction of solutions of linear generalized differential equations in a Banach space with the help of degenerate semigroups of linear bounded operators. The extensive bibliography on the indicated topics is contained in monographs [1, 2], which successfully complement each other. At the same time the theory of linear relations is covered insufficiently in the Russian mathematical literature. Let us draw attention to paper [3], in which linear relations on Hilbert space are considered.

Let us introduce principal notions of the theory of linear relations used below. We do not adhere to the terminology of [1, 2] (for example, we avoid the notion of the multivalued linear operator) and consider linear relations on one Banach space, as usual.

Let X and Y be complex Banach spaces. Every linear subspace $\mathcal{A} \subset X \times Y$ is called *a linear relation* between X and Y. If $\mathcal{A}$ is closed in $X \times Y$, then it is called *a closed linear relation.*

The subspace $D(\mathcal{A}) = \{x \in X \mid \exists y \in Y$ such that $(x, y) \in \mathcal{A}\}$ from X is called *the domain* of relation $\mathcal{A} \subset X \times Y$. The sets $\{y \in Y \mid (x, y) \in \mathcal{A}\}$, $\{x \in D(A) \mid (x, 0) \in \mathcal{A}\}$, $\{y \in Y \mid \exists x \in D(\mathcal{A})$ such that $(x, y) \in \mathcal{A}\} = \bigcup_{x \in D(A)} Ax$ are denoted by $\mathcal{A}x$ (where $x \in D(\mathcal{A})$), $Ker\, A$, $Im\, A$, respectively. Note that $D(\mathcal{A})$ and $Im\, \mathcal{A}$ is the projection of $\mathcal{A}$ on X and Y, respectively; $\mathcal{A}x = y + \mathcal{A}0\ \forall y \in \mathcal{A}x$.

The linear subspace $\mathcal{A} + \mathcal{B} = \{(x, y) \in X \times Y \mid x \in D(\mathcal{A}) \cap D(\mathcal{B}),\ y \in \mathcal{A}x + \mathcal{B}x\}$ is called *the sum* of linear relations $\mathcal{A}, \mathcal{B} \subset X \times Y$. So $D(\mathcal{A} + \mathcal{B}) = D(\mathcal{A}) \cap D(\mathcal{B})$, and the algebraic sum of sets $\mathcal{A}x, \mathcal{B}x$ is understood as $\mathcal{A}x + \mathcal{B}x$ for $x \in D(\mathcal{A}) \cap D(\mathcal{B})$.

Let Z be a Banach space. The linear subspace $\mathcal{B}\mathcal{A} = \{(x, z) \in X \times Z \mid \exists y \in D(\mathcal{B}) \subset Y$ such that $(x, y) \in \mathcal{A}, (y, z) \in \mathcal{B}\}$ is called *the product* of linear relations $\mathcal{A} \subset X \times Y, \mathcal{B} \subset Y \times Z$.

The relation $\mathcal{A}^{-1}$, which is defined by equality $\mathcal{A}^{-1} = \{(y, x) \in Y \times X\ \ (x, y) \in \mathcal{A}$, is called *the inverse relation* with respect to $\mathcal{A}$.

Each linear relation $\mathcal{A} \subset X \times Y$ is a graph of multivalued operator $\widetilde{\mathcal{A}} : D(\mathcal{A}) \subset X \to 2^Y$, where $\widetilde{\mathcal{A}}x = \mathcal{A}x \in 2^Y$. Further they are identified and the same symbol $\mathcal{A}$ is used for their notation.

Let us denote by $LR(X, Y)$ the set of closed linear relations from $X \times Y$; if $X = Y$, then we suppose that $LR(X) = LR(X, X)$. Moreover, set $LO(X, Y)$ of linear closed operators, acting from X to Y, is considered as a subspace of $LR(X, Y)$. Besides, $LR(X, Y)$ contains Banach space $Hom(X, Y)$ of linear bounded operators (homomorphisms), defined on X with values in Y. If $X = Y$, then $LO(X) = LO(X, X)$, and $End\, X$ is Banach algebra of linear bounded operators (endomorphisms), acting in X. Thus, $End\, X \subset LO(X) \subset LR(X)$.

Definition 1.1. The relation $\mathcal{A}$ from $LR(X, Y)$ is called *injective*, if $Ker\, \mathcal{A} = \{0\}$, and *surjective*, if $Im\, \mathcal{A} = Y$.

Definition 1.2. The relation $\mathcal{A}$ from $LR(X, Y)$ is called *continuously reversible*, if it is injective and surjective, simultaneously, and then $\mathcal{A}^{-1} \in Hom\,(Y, X)$.

The symbol I is used for the notation of the identity operator in any Banach space in the following definition and later on.

Definition 1.3. The set $\rho(\mathcal{A})$ of all $\lambda \in \mathbb{C}$ for which $(\mathcal{A} - \lambda I)^{-1} \in End\, X$, is called *the*

[1] The work was carried out with a financial support of The Russian Foundation for Basic Research, project 01-01-00408.

resolvent set of relation $\mathcal{A} \in LR(X)$. The set $\sigma(\mathcal{A}) = \mathbb{C} \setminus \rho(\mathcal{A})$ is called *the spectrum* of relation $\mathcal{A} \in LR(X)$.

The set $\rho(\mathcal{A})$ is open, set $\sigma(\mathcal{A})$ is closed.

Definition 1.4. The mapping

$$R(.,\mathcal{A}) : \rho(\mathcal{A}) \to End\, X,\ R(\lambda,\mathcal{A}) = (\mathcal{A} - \lambda I)^{-1},\quad \lambda \in \rho(\mathcal{A})$$

is called *the resolvent* of relation $\mathcal{A} \in LR(X)$.

The resolvent of relation $\mathcal{A} \in LR(X)$ is the pseudoresolvent in the generally accepted sense [4], and also $Ker\, R(\lambda_0,\mathcal{A}) = A\,0$, $Im\, R(\lambda_0,\mathcal{A}) = D(\mathcal{A})$, $\forall \lambda_0 \in \rho(\mathcal{A})$.

If $\mathcal{B} \in End\, X$ is the quasinilpotent operator, then $\sigma(\mathcal{B}^{-1}) = \varnothing$ (see § 2). To avoid problems connected with the possible emptiness of the spectrum of the relation let us use the following notion.

Definition 1.5. The subset of the extended complex plane $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, coinciding with $\sigma(\mathcal{A})$, if 1) $\mathcal{A}\,0 = \{0\}$, i. e. $\mathcal{A} \in LO(X)$; 2) the resolvent $R(\,.\,,\mathcal{A})$ of relation $\mathcal{A}$ admits the analytic extension into point ∞; 3) $\lim\limits_{|\lambda|\to\infty} R(\lambda,\mathcal{A}) = 0$, and coinciding with $\sigma(\mathcal{A}) \cup \{\infty\}$ in the opposite case is called *the extended spectrum* $\tilde{\sigma}(\mathcal{A})$ of relation $\mathcal{A} \in LR(X)$. The set $\widetilde{\rho}(\mathcal{A}) = \widetilde{\mathbb{C}} \setminus \widetilde{\sigma}(\mathcal{A})$ is called *the extended resolvent set* of relation $\mathcal{A} \in LR(X)$.

Note that if $X \neq \{0\}$, then $\widetilde{\sigma}(\mathcal{A}) \neq \varnothing\ \forall \mathcal{A} \in LR(X)$, besides, $\infty \in \widetilde{\sigma}(\mathcal{A})$, when $dim\, A\,0 \geq 1$, i. e. when $\mathcal{A} \in LR(X) \setminus LO(X)$.

The definition of the extended spectrum of the linear operator is found in monographs [4, 5]. Note that the results from §§ 2—5 demonstrate that the definition of the extended spectrum of the linear relation is of primary importance.

Urgency of investigation of linear relations is demonstarated by the examples of the problems given below. The presentation will be accompanied by the introduction of new notions and definitions.

1. If $\mathcal{A} \in LO(X)$ and $Ker\, \mathcal{A} \neq \{0\}$, then $\mathcal{A}^{-1}$ is a relation from $LR(X) \setminus LO(X)$.

2. Let $\mathcal{A} \in LO(X)$ be a linear operator with a nondense domain, i. e. $\overline{D(\mathcal{A})} \neq X$. Then the conjugate operator is not defined. Nevertheless one may define (see also [2. p. 1.5]) the conjugate relation $\mathcal{A}^* \subset X^* \times X^*$ to $\mathcal{A}$, where X^* is a dual Banach space to X, in a natural way:

$$A^* = \{(\eta,\xi) \in X^* \times X^* \mid \xi(y) = \eta(x)\, \forall (x,y) \in A\}. \quad (1.1)$$

It is clear that $\mathcal{A}^*0 = \{\eta \in X^* \mid \eta(x) = 0\, \forall x \in D(\mathcal{A})\} = D(\mathcal{A})^{\perp}$.

It is significant that this definition of $\mathcal{A}^*$ is appropriate for $\mathcal{A} \in LR(X) \setminus LO(X)$.

The definition of a conjugate linear relation was first given by von Neumann J. in [6], and his paper has given impetus to the development of the theory of linear relations, evidently.

3. Any pseudoresolvent $\mathcal{R} : U \subset \mathbb{C} \to End\, X$, defined on open set $U \subset \mathbb{C}$, is a resolvent of relation $\mathcal{A} = (\mathcal{R}(\lambda_0))^{-1} + \lambda_0 I$, where $\lambda_0 \in U$, and also $\rho(\mathcal{A}) \supset U$, and the definition of $\mathcal{A}$ does not depend on the choice of number λ_0 from U (see § 2).

4. The operator sequence $\mathcal{A}_n$ from $LO(X)$ is called *convergent*, if resolvent sets $\rho(\mathcal{A}_n)$, $n \geq 1$ have a nonempty intersection, besides, $\bigcap\limits_{n\geq 1} \rho(\mathcal{A}_n)$ contains an open connected set U, and for certain $\lambda_0 \in U$ sequence $R(\lambda_0,\mathcal{A}_n)$, $n \geq 1$ is fundamental with respect to the operator norm in $End\, X$. Then for any $\lambda \in U$ there exists $\lim\limits_{n\to\infty} R(\lambda,\mathcal{A}_n) = R(\lambda) \in End\, X$, and function $R : U \to End\, X$ is a pseudoresolvent, but it does not need to be a resolvent of a certain operator from $LO(X)$. With regard to p. 3 we conclude that limit $\mathcal{A}_0$ of the sequence of closed linear operators is, generally speaking, a linear relation.

5. The linear operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ is called admitting *a dense extension* (see [8, ch. III, § 1, p. 3]), if from conditions: 1) $\lim\limits_{n\to\infty} x_n = 0$, $x_n \in D(\mathcal{A})$; 2) there exists $\lim\limits_{n\to\infty} \mathcal{A}x_n = y$, it follows that $y = 0$. The equivalent definition: linear operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ admits a dense extension, if closure $\overline{\Gamma(\mathcal{A})}$ of its graph $\Gamma(\mathcal{A}) = \{(x,\mathcal{A}x) \in X \times X \mid x \in D(\mathcal{A})\}$ is a graph of a linear *operator*. Generally, the closure $\overline{\Gamma(\mathcal{A})}$ of the graph of operator $\mathcal{A}$ is *relation* $\mathcal{A} \in LR(X)$.

6. Let $\mathcal{A},\mathcal{B} \in Hom(X,Y)$. The function $\mathcal{P}(\lambda) = \mathcal{A} + \lambda\mathcal{B}$, $\lambda \in \mathbb{C}$ is called a linear bundle. It is known that many problems of mathematical physics are reduced to the study of the reversibility conditions of operators $\mathcal{P}(\lambda)$, $\lambda \in \mathbb{C}$. Linear bundles appear also after special transformations of polynomial bundles and S. G. Krein bundle, while the investigation of linear bundles is reduced in many cases to the study of spectral properties of relations $\mathcal{B}^{-1}\mathcal{A}$, $\mathcal{A}\mathcal{B}^{-1}$ from $LR(X)$, $LR(Y)$, respectively (see § 6).

7. Let us consider Cauchy problem

$$x(0) = x_0 \in X \quad (1.2)$$

for homogeneous linear differential equation

$$F\dot{x}(t) = Gx(t), \;\; t \in \mathbb{R}_+ = [0;+\infty) \quad (1.3)$$

with the pair of linear closed operators mapping from Banach space X to Banach space Y under condition $Ker\,F \neq \{0\}$.

In the questions of the solvability and of the construction of solutions to equation (1.3) two approaches are used. The first is founded on the spectral theory of ordered pairs of linear operators (see, for example, [9—12]). The second is based on the use of the linear generalized differential equation

$$\dot{y}(t) \in \mathcal{A}y(t),\; t \in \mathbb{R}_+,\; y(0) = y_0 \in D(\mathcal{A}), \quad (1.4)$$

where $\mathcal{A} \in LR(X)$ and it is written in the form $\mathcal{A} = F^{-1}G$. This technique is used in monograph [2] (see also [13]).

In this paper certain questions of the spectral theory of linear relations are considered, which are poorly dealt with in monographs [1, 2]. However, they are very useful in applications to the theory of generalized differential equations of form (1.4). In § 2 certain results concerning pseudoresolvents are contained, which one may obtain using linear relations, as well as theorems are proved about the spectral resolution of the linear relation and about the spectrum of the inverse relation. In § 3 isolation conditions of point ∞ in the extended spectrum of the linear relation are obtained. In § 4 with the help of ergodic theorems the description of the phase space for the generalized differential equation (1.4) is obtained, and also strongly continuous degenerate semigroups of operators with respect to linear relations with the use of certain analogs of Hille—Phillips—Yosida—Feller—Miyadera (HPYFM) theorem conditions [4] are constructed. In § 5 analytic degenerate semigroups are constructed with respect to sector linear relations. Applications of the spectral theory of linear relations to the spectral theory of ordered pairs of linear operators are given in § 6.

§ 2. On pseudoresolvents, the spectral resolution of the linear relation and spectrum of the inverse relation

The widest class of strongly continuous for $t > 0$ semigroups of bounded operators, which is considered in monograph [4], are semigroups of class E (see [4, definition 18.4.1]).

The pseudoresolvents, the construction of which was carried out in [4] with the help of Laplace transform of semigroups, give the total information about semigroups of class E. Sufficiently great attention is focused on the investigation of pseudoresolvents in [4, ch. 18, theorems 5.8.3—5.8.6, 5.9.1—5.9.3] as well as in a number of modern papers (see [1,2,7,14,15]). The spectral theory of linear relations may provide an essentially useful guide to the study of pseudoresolvents, since they are resolvents of linear relations, which were not considered in [4].

Let us describe this approach more explicitly. Let us appeal to a number of the well-known results and obtain them in a simple way and at times make them more precise.

Definition 2.1. The function $\mathcal{R} : \Omega \subset \mathbb{C} \to End\,X$ is called *a pseudoresolvent*, if for all $\lambda_1, \lambda_2 \in \Omega$ the equality (Hilbert identity)

$$\mathcal{R}(\lambda_1) - \mathcal{R}(\lambda_2) = (\lambda_1 - \lambda_2)\mathcal{R}(\lambda_1)\mathcal{R}(\lambda_2)$$

is fulfilled.

Note that definition 2.1 admits the case, when $\Omega = \{\lambda_0\}$ is a singleton, and then $\mathcal{R}(\lambda_0)$ may be an arbitrary endomorphism from algebra $End\,X$.

Definition 2.2. The pseudoresolvent $\mathcal{R}_{\max} : \Omega_{\max} \subset \mathbb{C} \to End\,X$ is called *a maximal extension* of pseudoresolvent $\mathcal{R} : \Omega \subset \mathbb{C} \to End\,X$, if it is the completion of any extension $\mathcal{R}$. Such pseudoresolvent is called *maximal*. Set $Sing\,\mathcal{R} = \mathbb{C}/\Omega_{max}$ is called *a singular set* of pseudoresolvent $\mathcal{R}$.

Definition 2.3. Let $Q \in End\,X$, $\lambda_0 \in \mathbb{C}$ and $\mathcal{R} : \Omega \subset \mathbb{C} \to End\,X$ be a pseudoresolvent. Operator Q will be called *embedded into* $\mathcal{R}$ *at point* λ_0, if $\lambda_0 \in \Omega$ and $Q = \mathcal{R}(\lambda_0)$.

Directly from definitions 2.1—2.3 it follows that the question of the embedding of a certain operator into the set of values of the pseudoresolvent may be reduced to the question of the construction of the maximal extension of the pseudoresolvent.

Theorem 2.1. *Every pseudoresolvent* $\mathcal{R} : \Omega \subset \mathbb{C} \to End\,X$ *has the unique maximal extension. It is a resolvent of a certain linear relation* $\mathcal{A}$, *and* $Sing\,\mathcal{R} = \sigma(\mathcal{A})$. *In particular, if* $Q \in End\,X$ *and* $\lambda_0 \in \mathbb{C}$, *then the unique maximal* *pseudoresolvent* $\mathcal{R}_0 : \Omega \subset \mathbb{C} \to End\,X$ *exists such that* $\lambda_0 \in \Omega$, *and operator* Q *is embedded into* $\mathcal{R}_0$ *at point* λ_0.

The statement of theorem 2.1 about the existence of the maximal extension was obtained in [4, theorem 5.8.6]. The statement about the embedding of the bounded operator into a certain pseudoresolvent is proved in [7, theorem 3.6]. Since the estimate

$$\|R(\lambda,\mathcal{A})\| \geq r(R(\lambda,\mathcal{A})) \geq (dist(\lambda,\sigma(\mathcal{A})))^{-1} \quad \forall\,\lambda \in \rho(\mathcal{A})$$

is valid (see corollary 2.1 of theorem 2.4), then theorem 2.1 contains also proposition 3.5 from [7] about the increase of the pseudoresolvent norm under the approach to $Sing\,\mathcal{R}$.

The notion of a pseudoresolvent singular set was introduced in [15]. By virtue of theorem 2.1 it coincides with the linear relation spectrum, the resolvent of which is the extension of the pseudoresolvent under investigation.

The following theorem defines more exactly statement 5.8.4 from [4], and it easily arises from theorem 2.1.

In its conditions let us denote by symbol $Sp\,\mathfrak{A}$ the spectrum of the commutative Banach algebra $\mathfrak{A}$ with the unity, i. e. $Sp\,\mathfrak{A}$ is a compact topological space of nonzero continuous complex homomorphisms of algebra $\mathfrak{A}$, $\hat{a} : Sp\,\mathfrak{A} \to \mathbb{C}$, $\hat{a}(\chi) = \chi(a)$, $\chi \in Sp\,\mathfrak{A}$ is Gelfand transform of element a from $Sp\,\mathfrak{A}$ (see [16]).

Theorem 2.2. *Let* $\mathcal{R} : \Omega \subset \mathbb{C} \to End\,X$ *be a maximal pseudoresolvent and be the least closed subalgebra from Banach algebra* $End\,X$, *which contains all operators* $\mathcal{R}(\lambda)$, $\lambda \in \Omega$ *and operator* I. *Then its spectrum* $Sp\,\mathfrak{A}$ *is homeomorphic to the extended spectrum* $\tilde{\sigma}(\mathcal{A})$ *of linear relation* $\mathcal{A} \in LR(X)$, *for which function* $\mathcal{R} : \Omega = \rho(\mathcal{A}) \to End\,X$ *is the resolvent. Further, homeomorphism* $\alpha : Sp\,\mathfrak{A} \to \tilde{\sigma}(\mathcal{A})$ *exists, for which*

$$\widehat{R(\lambda,\mathcal{A})}(\chi) = \frac{1}{\lambda - \alpha(\chi)},\ \chi \in Sp\,\mathfrak{A},\ \lambda \in \rho(\mathcal{A}).$$

Besides, $\alpha(\chi_\infty) = \infty$ *for the character* $\chi_\infty \in Sp\,\mathfrak{A}$, *defined by conditions*: $R(\lambda,\mathcal{A}) \in Ker\,\chi_\infty\ \forall\,\lambda \in \rho(\mathcal{A})$, $\chi_\infty(I) = 1$.

Due to theorem 2.1 the definitions, introduced further, and the statements, proved on their basis, are quite correct.

Definition 2.4. The closed linear subspace $X_0 \subset X$ is called *invariant* for relation $\mathcal{A} \in LR(X)$ with nonempty $\rho(\mathcal{A})$, if X_0 is invariant with respect to all operators $R(\lambda,\mathcal{A})$, $\lambda \in \rho(\mathcal{A})$. *The restriction of relation* $\mathcal{A} \in LR(X)$ on subspace X_0 is called relation $\mathcal{A}_0 \in LR(X)$, the resolvent of which is the restriction $R_0 : \rho(\mathcal{A}) \to End\,X_0$, $R_0(\lambda) = R(\lambda,\mathcal{A}) \upharpoonright X_0$, $\lambda \in \rho(\mathcal{A})$ of resolvent $R(\,.\,,\mathcal{A}) : \rho(\mathcal{A}) \to End\,X$ on X_0 and it is denoted by $\mathcal{A}_0 = \mathcal{A} \upharpoonright X_0$.

Definition 2.5. Let

$$X = X_0 \oplus X_1 \tag{2.1}$$

be a direct sum of invariant with respect to $\mathcal{A} \in LR(X)$ subspaces, $\mathcal{A}_0 = \mathcal{A} \upharpoonright X_0$, $\mathcal{A}_1 = \mathcal{A} \upharpoonright X_1$. Then relation $\mathcal{A}$ is called *a direct sum of relations* $\mathcal{A}_0$ and $\mathcal{A}_1$, and it is written as

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1. \tag{2.2}$$

In addition, $\mathcal{A}\,0 = \mathcal{A}_0\,0 \oplus \mathcal{A}_1\,0$, and equalities (2.1), (2.2) mean that set $\mathcal{A}x$ for every $x \in D(\mathcal{A})$ is defined by formulae

$$\mathcal{A}x = \mathcal{A}_0 x_0 + \mathcal{A}_1 x_1,\ x = x_0 + x_1,$$

where $x_i \in D(\mathcal{A}_i) \subset X_i$, $i = 0,1$ and $\mathcal{A}x$ is an algebraic sum of sets $\mathcal{A}_0 x_0$, $\mathcal{A}_1 x_1$.

Lemma 2.1. *If for relation* $\mathcal{A} \in LR(X)$ *equalities* (2.1), (2.2) *take place, then* $\tilde{\sigma}(\mathcal{A}) = \tilde{\sigma}(\mathcal{A}_0) \cup \tilde{\sigma}(\mathcal{A}_1)$, *where* $\mathcal{A}_i$ *is the restriction* $\mathcal{A}$ *on* X_i, $i = 0,1$.

Lemma 2.2. *Let* $\mathcal{A} \in LR(X)$. *Then* $\infty \notin \tilde{\sigma}(\mathcal{A}) \Longleftrightarrow \mathcal{A} \in End\,X$.

Note that in [1] the condition $\infty \notin \tilde{\sigma}(\mathcal{A})$ for relation $\mathcal{A} \in LR(X)$ means, in the definition, that $0 \notin \sigma(\mathcal{A}^{-1})$.

Theorem 2.3. *Let* $\mathcal{A} \in LR(X)$ *and its extended spectrum* $\tilde{\sigma}(\mathcal{A})$ *be represented in the form*

$$\tilde{\sigma}(\mathcal{A}) = \sigma_0 \cup \sigma_1, \tag{2.3}$$

where σ_0 *is a compact set from* $\mathbb{C}$, σ_1 *is a closed set from* $\tilde{\mathbb{C}}$ *and* $\sigma_0 \cap \sigma_1 = \varnothing$. *Then expansions* (2.1), (2.2) *exist, in which invariant with respect to* $\mathcal{A}$ *closed subspaces* X_0, X_1 *and restrictions* $\mathcal{A}_0 = \mathcal{A} \upharpoonright X_0$, $\mathcal{A}_1 = \mathcal{A} \upharpoonright X_1$ *possess the following properties*:

1) $\mathcal{A}_0 \in End\,X_0$, $\tilde{\sigma}(\mathcal{A}_0) = \sigma(\mathcal{A}_0) = \sigma_0$;

2) $\mathcal{A}_1 0 = \mathcal{A}0 = Ker\,R(\,.\,,\mathcal{A}) = Ker\,R(\,.\,,\mathcal{A}_1) \subset X_1$, $D(\mathcal{A}) = X_0 \oplus D(\mathcal{A}_1)$, $\tilde{\sigma}(\mathcal{A}_1) = \sigma_1$.

Theorem 2.4. *If* $\mathcal{A} \in LR(X)$, *then the extended spectrum* $\tilde{\sigma}(\mathcal{A}^{-1})$ *of the inverse relation* $\mathcal{A}^{-1} \in LR(X)$ *to* $\mathcal{A}$ *is represented in the form* $\tilde{\sigma}(\mathcal{A}^{-1}) = \{\lambda^{-1} \mid \lambda \in \tilde{\sigma}(\mathcal{A})\}$.

Corollary 2.1. *If* $\mathcal{A} \in LR(X)$ *and* $\mu \in \rho(\mathcal{A})$, *then* $\sigma(R(\mu,\mathcal{A})) = \{(\mu - \lambda)^{-1} \mid \lambda \in \tilde{\sigma}(\mathcal{A})\}$.

The conclusion of corollary 2.1 is obtained in [1, theorem V.4.2].

Note that the definition of the linear relation spectrum was introduced by A. Favini, A. Yagi in [17], however, they do not use the notion of the linear relation extended spectrum neither in [17], nor in monograph [2]. The definition of the extended spectrum for linear relations on normed spaces was introduced in monograph [1] by R. Cross, but, essentially, it was little used.

Corollary 2.2. *For $\mathcal{A} \in LR(X)$ the following conditions are equivalent:*

$$1)\ \mathcal{A} \in End\, X;\ \ 2)\ \infty \notin \widetilde{\sigma}(\mathcal{A});\ \ 3)\ 0 \notin \sigma(\mathcal{A}^{-1}).$$

Corollary 2.3. *For $\mathcal{A} \in LR(X)$ equality $\widetilde{\sigma}(\mathcal{A}) = \{\infty\}$ is valid iff $\mathcal{A}^{-1} \in End\, X$ is a quasinilpotent operator.*

Corollary 2.4. *If $\lambda \in \rho(\mathcal{A}), \lambda \neq 0,$ then the following equalities take place*

$$(\mathcal{A}^{-1} - \lambda^{-1} I)^{-1} = -\lambda I - \lambda^2 (\mathcal{A} - \lambda I)^{-1},$$
$$(\mathcal{A} - \lambda I)^{-1} = -\lambda^{-1} I - \lambda^{-2} (A^{-1} - \lambda^{-1} I)^{-1}. \tag{2.4}$$

Corollary 2.5. *If $\mathcal{A} = B^{-1}$, where B is a quasinilpotent operator from $End\, X$, then $\widetilde{\sigma}(\mathcal{A}) = \{\infty\}$ and $(\mathcal{A} - \lambda I)^{-1} = B + \lambda B^2 + \lambda^2 B^3 + \dots$. In particular, $R(.,\mathcal{A})$ is a polynomial, if B is the nilpotent operator from $End\, X$.*

Theorem 2.4 and correlations (2.4) allow us to state that for the extended spectrum of linear relations all points of the extended complex plane $\widetilde{\mathbb{C}}$, including ∞, have the same rights in a sense. If point ∞ is contained in the extended spectrum of the linear closed operator and is isolated there, then it may only be an essential singularity of its resolvent (see [4, theorem 5.9.4]).

Let Δ be an open set from the extended complex plane $\widetilde{\mathbb{C}}$ containing the extended spectrum $\widetilde{\sigma}(\mathcal{A})$ of the relation. The algebra of complex functions, defined and analytic on Δ, will be denoted by symbol $\mathfrak{F}(\Delta)$. Let γ be a certain closed Jordan curve, surrounding $\widetilde{\sigma}(\mathcal{A})$, and function $f \in \mathfrak{F}(\Delta)$ be such that integral $\int_\gamma f(\lambda) R(\lambda, \mathcal{A}) d\lambda$ converges absolutely. Then formula

$$f(\mathcal{A}) = \delta f(\infty) I - \frac{1}{2\pi i} \int_\gamma f(\lambda) R(\lambda, \mathcal{A}) d\lambda \tag{2.5}$$

defines the bounded operator from algebra $End\, X$, where $\delta = 1$ or $\delta = 0$ depending on whether $\lambda = \infty$ is inside γ or outside of γ. Moreover, it belongs to commutative subalgebra $\mathfrak{A}$, introduced before theorem 2.2. This fact allows us to obtain the following statement:

Theorem 2.5. *For $\mathcal{A} \in LR(X)$ the equality $\sigma(f(\mathcal{A})) = f(\widetilde{\sigma}(\mathcal{A})) = \{f(\lambda) \mid \lambda \in \widetilde{\sigma}(\mathcal{A})\}$ takes place.*

§ 3. Compactness conditions of the linear relation spectrum

In the remaining part of the paper it is supposed that the following condition of *nonsingularity* of linear relations is fulfilled.

Assumption 3.1. Resolvent set $\rho(\mathcal{A})$ of linear relation $\mathcal{A} \in LR(X)$ is not empty.

Immediately from the definition of the inverse relation and from the properties of linear relations formulated in § 2 (see also the properties of relations, enumerated in § 6) follows

Lemma 3.1. *Let $\mathcal{A} \in LR(X)$. Independent of the choice $\lambda_0 \in \rho(\mathcal{A})$ equalities*

$$Ker\,(R(\lambda_0, \mathcal{A}))^k = A^k\, 0,$$
$$Im(R(\lambda_0, \mathcal{A}))^k = D(\mathcal{A}^k),\ k \in N$$

are valid.

This lemma ensures the correctness of notations $Ker\, R^k$ and $Im\, R^k, k \in N$ for the degrees of the resolvent of relation $\mathcal{A} \in LR(X)$.

Definition 3.1. Relation $\mathcal{A} \in LR(X)$. is said to possess the property of *degrees stability in infinity*, if the number $m \in \mathbb{N}$ exists such that

$$\mathcal{A}^{m-1}\, 0 \subset \mathcal{A}^m\, 0 = \mathcal{A}^{m+1}\, 0,$$
$$D(\mathcal{A}^{m-1}) \supset D(\mathcal{A}^m) = D(\mathcal{A}^{m+1}), \tag{3.1}$$

where inclusions are strict. The number m is called *the order* of the stability.

Note that for $m = 1$ it is assumed that $\{0\} \subset \mathcal{A}\, 0 = \mathcal{A}^2\, 0,\ X \supset D(\mathcal{A}) = D(\mathcal{A}^2)$.

Assumption 3.2. Relation $\mathcal{A} \in LR(X)$. possesses the property of degrees stability in infinity of the order m.

Theorem 3.1. *Let $m \geq 2$ be a natural number. For relation $\mathcal{A} \in LR(X) \setminus LO(X)$ the following conditions are equivalent:*

1) point ∞ is the pole of function $R(.,\mathcal{A})$ of the order $m-2$ for $m \geq 3$, ∞ is the removable singularity of function $R(.,\mathcal{A})$ for $m = 2$;

2) Banach space X is represented in the form of the direct sum $X = X_0 \oplus X_\infty$ of invariant with respect to $\mathcal{A}$ closed subspaces $X_0 = D(A^m)$, $X_\infty = \mathcal{A}^m\, 0$, and also the restriction $\mathcal{A}_0$ of relation $\mathcal{A}$ on X_0 belongs to $End\, X_0$, $\sigma(\mathcal{A}_0) = \sigma(\mathcal{A})$, and, besides, $\widetilde{\sigma}(\mathcal{A}_\infty) = \{\infty\}$, $\mathcal{A}_\infty^{-1} \in End\, X_\infty$, $(\mathcal{A}_\infty^{-1})^m = 0$

for the restriction $\mathcal{A}_\infty$ *of relation* $\mathcal{A}$ *on* X_∞ *and* $(\mathcal{A}_\infty^{-1})^{m-1} \neq 0$;

3) *conditions of assumption* 3.2 *are fulfilled*.

Results from [18, ch. 6] and [19, theorem 2.2] are used in the proof.

Let us introduce into consideration eigenvectors and adjoint vectors of linear relations, corresponding to point ∞. Here the results of theorems 2.4 and 3.1 will be taken into account.

Definition 3.2. An arbitrary nonzero vector x_0 from $\mathcal{A}\,0 \subset X$ is called *the eigenvector* of relation $\mathcal{A} \in LR(X)$, corresponding to point ∞. Vector $x_1 \in X$ is called *the root vector* of relation $\mathcal{A}$, corresponding to point ∞, if the number $k \in \mathbb{N}$ exists such that $x_1 \in \mathcal{A}^k\,0$. The number k from $\mathbb{N}$ is called *the height* of the root vector x_1 if $x_1 \in \mathcal{A}^k\,0 \setminus \mathcal{A}^{k-1}\,0$.

Immediately from definition 3.2 it follows that the closed subspace $\mathcal{A}^k\,0 = Ker\,(\mathcal{A}^{-1})^k$ consists of root vectors of relation $\mathcal{A}$, corresponding to point ∞, with the height not exceeding k.

Definition 3.3. The relation $\mathcal{A} \in LR(X)$ is said to have the finite Jordan chain $x_0, x_1, \ldots, x_{k-1}$ of the height k, corresponding to point ∞, if x_0 is the eigenvector for $\mathcal{A}$, corresponding to point ∞, and x_i, $2 \le i \le k-1$ are root vectors, corresponding to the same point, for which the following correlations

$$x_0 \in \mathcal{A}\,0,\ x_i \in \mathcal{A}x_{i-1},\ 1 \le i \le k-1,\ x_{k-1} \notin \mathcal{A}^k\,0$$

take place (and, consequently, every vector x_i, $0 \le i \le k-1$ has the height i). Vectors $x_1, \ldots, x_{k-1}$ are called *adjoint* to eigenvector x_0.

Lemma 3.2. *The relation* $\mathcal{A} \in LR(X)$ *has the finite Jordan chain* $x_0, \ldots, x_{k-1}$ *of the height* k, *corresponding to point* ∞, *iff for certain* $\lambda_0 \in \rho(\mathcal{A})$ *(and hence for all* $\lambda_0 \in \rho(\mathcal{A})$*) equalities*

$$R(\lambda_0, \mathcal{A})x_0 = 0,\ x_0 \in \mathcal{A}\,0,\ R(\lambda_0, \mathcal{A})x_i = x_{i-1},$$
$$0 \le i \le k-1$$

are valid, and vector x is absent such that $R(\lambda_0, \mathcal{A})x = x_{k-1}$.

Definition 3.4. The relation $\mathcal{A} \in LR(X)$ is called *Fredholm in infinity*, if $D(\mathcal{A})$ is a closed subspace in X, and, besides, $\mathcal{A}\,0$, $X/D(\mathcal{A})$ are finitedimensional linear spaces. The number $ind\,\mathcal{A} = dim\,\mathcal{A}\,0 - dim\,(X/D(\mathcal{A})$ is called *the index* of Fredholm relation $\mathcal{A}$, corresponding to point ∞.

Directly from definition 3.4 it follows that relation $\mathcal{A}$ is Fredholm in infinity iff operator $R(\lambda_0, \mathcal{A})$, $\lambda_0 \in \rho(\mathcal{A})$ is Fredholm, and their indices coincide.

The following statement arises from theorem 3.1, definitions 3.2—3.4 and lemma 3.2, and it deciphers the notions contained in them. Let us take into account that if the relation index is equal to zero, then indices of all its degrees are the same.

Theorem 3.2. *Let* $m \in \mathbb{N}$. *For Fredholm in infinity relation* $\mathcal{A} \in LR(X) \setminus LO(X)$ *of zero index the following conditions are equivalent:*

1) *all Jordan chains of relation* $\mathcal{A}$, *corresponding to point* ∞, *have the height which does not exceed number* $m \in \mathbb{N}$, *moreover, Jordan chain exists with height m;*

2) $\mathcal{A}^{m-1}\,0 \subset \mathcal{A}^m\,0 = \mathcal{A}^{m+1}\,0$;

3) $D(\mathcal{A}^{m-1}) \supset D(\mathcal{A}^m) = D(\mathcal{A}^{m+1})$;

4) *point* ∞ *is the pole of resolvent* $R(\lambda, \mathcal{A})$, $\lambda \in \rho(\mathcal{A})$ *of relation* $\mathcal{A}$ *of the order* $m-2$, *if* $m \le 3$, *and point* ∞ *is the removable singularity, if* $m = 2$.

Corollary 3.1. *If* $\mathcal{A} \in LR(X) \setminus LO(X)$, X *is a finite-dimensional space, then* $\widetilde{\sigma}(\mathcal{A})$ *consists of a finite set of points, the number of which does not exceed* $n = \dim X$, *and also for relation* $\mathcal{A}$ *the following spectral resolution*

$$\mathcal{A} = \sum_{i=1}^{m} \lambda_i P_i + Q + \mathcal{A}_\infty,\ m+1 \le n$$

takes place, where $P_i \in End\,X$ *are Riesz projectors, constructed on singletons* $\{\lambda_i\}$, $1 \le i \le m$, $\sigma(\mathcal{A}) = \{\lambda_1, \ldots \lambda_m\}$, $\widetilde{\sigma}(\mathcal{A}_\infty) = \{\infty\}$, Q, $\mathcal{A}_\infty^{-1}$ *are nilpotent operators from algebra* $End\,X$, *which are commutative between themselves and with projectors* P_i, $1 \le i \le m$.

§ 4. On certain analogs of conditions Hille—Phillips—Yosida—Feller—Miyadera for linear relations

Let $\mathcal{A}$ be a relation from $LR(X)$. Let us consider Cauchy problem for the generalized differential equation

$$\dot{x}(t) \in \mathcal{A}x(t),\ t \in \mathbb{R}_+ = [0; +\infty), \qquad (4.1)$$

$$x(0) = x_0 \in D(\mathcal{A}). \qquad (4.2)$$

Differentiable function $x : \mathbb{R}_+ \to X$, for which $x(0) = x_0$, $x(t) \in D(\mathcal{A})\ \forall\, t \ge 0$, is called *the solution* to Cauchy problem (4.1)—(4.2), if it satisfies inclusion (4.1).

Definition 4.1. The closure in X of initial conditions set of the form (4.2), for which the solution to problem (4.1)—(4.2) exists, is called *the phase space* of the generalized differential equation (4.1), and it is denoted by symbol $\Phi(\mathcal{A})$.

In this paragraph the relations are considered, for which ∞ is not necessarily an isolated point in the extended spectrum. With the help of ergodic theorems the subspaces are formed, containing the phase space for the generalized differential equations, and thereupon with the help of certain analogs of Hille—Phillips—Yosida—Feller—Miyadera (HPYFM) theorem conditions degenerate semigroups of linear bounded operators are constructed for linear relations.

It is provided, as before, that assumption 3.1 holds.

Definition 4.2. Let $m \in \mathbb{N}$. The degree m of resolvent of relation $\mathcal{A} \in LR(X)$ is said to possess the property of *the minimal growth in infinity*, if sequence $\{\lambda_n\} \subset \rho(\mathcal{A})$ exists such that

$$1) \lim_{n\to\infty} |\lambda_n| = \infty;$$

$$2) \sup_{n1} |\lambda_n^m| \cdot \| (R(\lambda_n, \mathcal{A}))^m \| \} < \infty. \tag{4.3}$$

Assumption 4.1. The resolvent of relation $\mathcal{A} \in LR(X)$ satisfies conditions (4.3) from definition 4.2.

Theorem 4.1. *If for relation $\mathcal{A} \in LR(X)$ assumption* 3.2 *is fulfilled, then for it assumption* 4.1 *is fulfilled too.*

Lemma 4.1. *If assumption* 4.1 *is fulfilled, then lengths of all Jordan chains of relation $\mathcal{A} \in LR(X)$, corresponding to point ∞, do not exceed m, and all chains lie in $X_\infty = \mathcal{A}^m 0$.*

Under the conditions of assumption 4.1 let us introduce into consideration the bounded sequence of operators from algebra $End\,X$ of the form

$$\mathcal{A}_n = I - (-\lambda_n R(\lambda_n, \mathcal{A}))^m, \; n \in \mathbb{N} \tag{4.4}$$

and the closed subspace

$$\widetilde{X} = \{x \in X : \exists \lim_{n\to\infty} \mathcal{A}_n x\}.$$

For the construction of the phase space $\Phi(\mathcal{A})$ of the generalized differential equation (4.1) let us use ergodic theorems from paper [20], applied to the consequence A_n. At first let us formulate certain notions and results from [20], used here (not in the most general form).

Let $\mathfrak{A}$ be the least closed subalgebra from Banach algebra $End\,X$, containing all operators $R(\lambda;\mathcal{A})$, $\lambda \in \rho(\mathcal{A})$ and the identity operator I. Then $\mathfrak{A}$ is a commutative Banach algebra with the unity and sequence (A_n) belongs to $\mathfrak{A}$. Let $m \in \mathbb{N}$. Let us consider the least closed ideal $\mathcal{J} = \mathcal{J}_m$ from algebra $\mathfrak{A}$, containing operators $(R(\lambda;\mathcal{A}))^m$, $\lambda \in \rho(\mathcal{A})$.

Definition 4.3. The bounded sequence of linear operators (A_n) from algebra $\mathfrak{A}$ is called $\mathcal{J}$*-sequence*, if the following two conditions are fulfilled:

$$1) \lim_{n\to\infty} \| A_n F \| = 0 \quad \forall F \in \mathcal{J};$$

$$2) A_n x - x \in \overline{\mathcal{J}x} = \overline{\{Fx;\, F \in \mathcal{J}\}} \quad \forall x \in X.$$

Let (A_n) be $\mathcal{J}$-sequence. By symbol $Erg(X,(A_n))$ we denote (closed) subspace

$$Erg(X,(A_n)) = \{x \in X : \exists \lim_{n\to\infty} A_n x\},$$

and it is called *the ergodic* subspace, corresponding to $\mathcal{J}$-sequence (A_n).

Lemma 4.2. *Under the conditions of assumption* 4.1 *sequence (A_n) is $\mathcal{J}$-sequence, and, hence, $\widetilde{X} = Erg(X,(A_n))$.*

Since subspace $\widetilde{X}$ is invariant with respect to all operators $R(\lambda,\mathcal{A})$, $\lambda \in \rho(\mathcal{A})$, then it is invariant with respect to relation $\mathcal{A}$, and so one may consider the restriction $\widetilde{\mathcal{A}}$ of relation $\mathcal{A}$ on $\widetilde{X}$ (see definition 2.4).

The following statement arises from [20, lemma 1] and is the concrete realization of the properties, formulated there.

Theorem 4.2. *Under the conditions of assumption* 4.1 *subspace $\widetilde{X}$ admits the expansion into the direct sum*

$$\widetilde{X} = X_0 \oplus X_\infty \tag{4.5}$$

of two closed invariant with respect to $\mathcal{A}$ subspaces X_0, X_∞, and also $X_0 = D(\mathcal{A}^m)$, $X_\infty = \mathcal{A}^m 0$, and the corresponding expansion of relation $\widetilde{\mathcal{A}} \in LR(\widetilde{X})$

$$\widetilde{\mathcal{A}} = \mathcal{A}_0 \oplus \mathcal{A}_\infty \tag{4.6}$$

possesses properties: $\widetilde{\sigma}(\mathcal{A}_\infty) = \{\infty\}$, $(\mathcal{A}_\infty^{-1})^m = 0$, $\mathcal{A}_0 : D(\mathcal{A}_0) \subset X_0 \to X_0$ is a linear closed operator with the spectrum $\sigma(\mathcal{A}_0) = \sigma(\widetilde{\mathcal{A}}) = \sigma(\mathcal{A})$ and with the dense in X_0 domain $D(\mathcal{A}_0^m)$ of operator $\mathcal{A}_0^m$.

Remark 4.1. The projector P_0, which realizes the expansion (4.5) of space $\widetilde{X}$, is defined by correlations

$$P_0 x = \lim_{n\to\infty}(I - A_n)x = \lim_{n\to\infty}(-\lambda_n R(\lambda_n, \mathcal{A}))^m x,$$

$$x \in \widetilde{X},\; Im\,P_0 = X_0,\; Ker\,P_0 = X_\infty,$$

and it does not depend on the choice of sequence (λ_n) from $\rho(\mathcal{A})$, satisfying conditions of assumption 4.1 (see [20, lemma 2]). Besides, $D(\mathcal{A}_0^k)$ is dense in X_0 for every $k \in \mathbb{N}$ (see [20]).

Corollary 4.1. *Under the conditions of assumption* 4.1 *relation* $\mathcal{A} \in LR(X)$ *is a linear operator, if domain* $D(\mathcal{A}^m)$ *of relation* $\mathcal{A}^m$ *is dense in X.*

The following statement arises from [20, theorem 1].

Theorem 4.3. *Let assumption* 4.1 *be fulfilled. In order that* $\widetilde{X} = X$, *it is necessary and sufficient that the vectors from subspace* $\mathcal{A}^m 0$ *should separate functionals from subspace* $(\mathcal{A}^*)^m 0$ *of the dual to X Banach space* X^* ($\mathcal{A}^* \subset X^* \times X^*$ *is the conjugate to* $\mathcal{A}$ *linear relation; see § 1, p. 2).*

In particular, $\widetilde{X} = X$, *if one of the following conditions is fulfilled:*

1) *X is a reflexive Banach space;*

2) $R(\lambda_0, \mathcal{A}) \in End\, X$ *is a weakly compact operator for certain* $\lambda_0 \in \rho(\mathcal{A})$;

3) $dim\, \mathcal{A}^m 0 = dim\, (\mathcal{A}^*)^m 0 < \infty$.

Note that the statement of theorem 4.2 for a reflexive Banach space and for $m = 1$ is given in [2, p. 1.3] and in [13].

Corollary 4.2. *If relation* $\mathcal{A} \in LR(X)$ *has a compact resolvent under the conditions of assumption* 4.1, *then* $X = X_0 \oplus X_\infty = \mathcal{A}^m 0 \oplus \overline{D(\mathcal{A}^m)}$.

Assumption 4.2. There exist such numbers $M > 0, \omega \in R, m \in N$, that for all $\lambda \in C$ with $Re\lambda > \omega$ and for all $n \in N$ estimates

$$(R(\lambda, \mathcal{A}))^{mn} \frac{M}{(Re\,\lambda - \omega)^{mn}}, \ n \in \mathbb{N} \tag{4.7}$$

take place.

Let us carry out the construction of the phase space $\Phi(\mathcal{A})$ and degenerate semigroups of linear operators, with the help of which the solutions to problem (4.1)—(4.2) are defined. The constructions are realized under the conditions of assumption 4.2 and for $dim\, A\, 0 \geq 1$, i. e. $\mathcal{A} \in LR(X) \setminus LO(X)$. Assumption 4.2 implies assumption 4.1, so according to lemma 4.1 one may consider an ergodic subspace $\widetilde{X} = Erg\,(X, (\mathcal{A}_n))$, constructed with the help of the bounded sequence $(\mathcal{A}_n) \in End\, X$. It is defined by formula (4.4), where (λ_n) is an arbitrary sequence from $\mathbb{R}_+ \cap \rho(\mathcal{A})$ with the property $\lim\limits_{n\to\infty} \lambda_n = \infty$. Thus, the statement of theorem 4.2 about the decomposition of subspace $\overline{\widetilde{X}}$ is valid. Besides, subspaces $X_\infty = \mathcal{A}^m 0$, $X_0 = \overline{D(\mathcal{A}^m)}$ are invariant for relation $\mathcal{A}$. For the restriction $\mathcal{A}_0 = \mathcal{A} \restriction X_0 \in LO(X_0)$ of relation $\mathcal{A}$ on X_0 assumption 4.2 remains fulfilled. It allows us to construct on X_0 semigroup $\{T_0(t);\, t \geq 0\}$ of class C_0 with the generator $\mathcal{A}_0$, having, according to theorem 4.2, the dense domain $D(\mathcal{A}_0)$ in X_0. For the construction of such a semigroup let us use the analog of Yosida approximation (see [4, theorem 12.3.1]) of the form:

$$A_n^0 = (-\lambda_n/m)(I - (-\lambda_n R(\lambda_n, \mathcal{A}_0))^m) \in End\, X_0,\ n \geq 1.$$

Lemma 4.3. *Under the conditions of assumption* 4.1 *for every* $\mu \in \rho(\mathcal{A})$ *the estimate*

$$\left\| (A_n^0 - (-\lambda_n R(\lambda_n, \mathcal{A}_0))^m \mathcal{A}_0)(R(\mu, \mathcal{A}_0))^m \right\| \leq const \cdot |\lambda_n|^{-1}, \quad n \geq 1 \tag{4.8}$$

is valid.

Theorem 4.4. *Let for relation* $\mathcal{A} \in LR(X)$ *assumption* 4.2 *be fulfilled and* $dim\, A\, 0 \geq 1$. *Then*

$$\Phi(\mathcal{A}) \cap \widetilde{X} = \overline{D(\mathcal{A}^m)} = X_0,$$

and the unique degenerate semigroup of operators $\{\widetilde{T}(t); t \geq 0\} \subset End\, \widetilde{X}$ *exists, the generator of which is relation* $\widetilde{\mathcal{A}} \in LR(\widetilde{X})$, *defined by equalities* $\widetilde{\mathcal{A}} = \mathcal{A}_0$ *on* X_0, $D(\widetilde{\mathcal{A}}) = X_0 \cap D(\mathcal{A})$, $\widetilde{\mathcal{A}}0 = X_\infty$. *Semigroup* $\{\widetilde{T}(t);\, t \geq 0\}$ *possesses the following properties:*

1) *its restriction* $\{T_0(t); 0\} \subset End\, X_0$ *on* X_0 *is a semigroup of class* C_0, *and any solution* $x : \mathbb{R}_+ \to X$ *to problem* (4.1)—(4.2) *with* $x_0 \in D(\mathcal{A}_0) \subset X_0$ *is written as* $x(t) = T_0(t)x_0$, $t \geq 0$;

2) $\widetilde{T}(0) \in End\, \widetilde{X}$ *is a projector on subspace* X_0, *parallel to* X_∞.

If vectors from subspace $X_\infty = \mathcal{A}^m 0 \subset X$ *separate functionals from subspace* $X_\infty^* = (\mathcal{A}^*)^m 0 \subset X^*$ *(for example, if one of three conditions of theorem* 4.3 *is fulfilled), then* $\widetilde{X} = X$, *and* $\Phi(\mathcal{A}) = X_0$.

Remark 4.2. Statements of theorem 4.4 for $m = 1$ are contained in monograph [2, ch. II]. If $\mathcal{A}$ is a linear relation on finite-dimensional space X, and also $\mathcal{A}^2 0 \neq \mathcal{A}\, 0$, then results from [2] are inapplicable even in this case. The expansion $X = \overline{D(\mathcal{A})} \oplus \mathcal{A}\, 0$ was obtained in [2] only for a reflexive Banach space. For $m > 1$ generalized differential equations of the form (4.1) are considered in [13] by the *n-integrated* semigroups method. However, principal results are announced in [13] under a priori assumption about the existence of the expansion $X = \overline{D(\mathcal{A}^m)} \oplus \mathcal{A}^m 0$; its presence was marked for a reflexive Banach space X under the condition of $m = 1$.

Corollary 4.3. *Let for relation* $\mathcal{A} \in LR(X)$, *satisfying assumption* 4.1, *numbers* $M > 0, \omega \in \mathbb{R}$

exist such that for all $x \in X_0$, $\lambda \in \mathbb{C}$ *with* $Re\,\lambda > \omega$ *and all* $n \in \mathbb{N}$ *estimates*

$$\| (R(\lambda, A))^n x \| \le \frac{M \| x \|}{(Re\,\lambda - \omega)^n}$$

take place. Then all statements of theorem 4.4 *are valid.*

Directly from theorem 4.4 follows

Theorem 4.5. *If for linear operator* $A \in LO(X)$ *with* $\overline{D(A)} = X$ *conditions of assumptions* 4.2 *are fulfilled, then* A *is a generator of a semigroup of class* C_0.

From theorems 3.1 (condition 2)), 4.1 and 4.4 arises

Theorem 4.6. *If relation* $\mathcal{A} \in LR(X)$ *satisfies one of the condition of theorem* 3.1, *then* $\Phi(\mathcal{A}) = X_0$, *and every solution* $x : \mathbb{R}_+ \to X$ *to problem* (4.1)—(4.2) *with* $x_0 \in X_0$ *is defined with the help of the analytic group of operators* $\{\exp A_0 t, t \in R\}$ *and it is written in the form* $x(t) = (\exp \mathcal{A}_0 t) x_0$, $t \in R$.

Note that semigroup $\{\widetilde{T}(t); t \ge 0\}$, constructed in theorem 4.4 according to expansion (3.2) of space X, has the form $\widetilde{T}(t) = \exp \mathcal{A}_0 t \oplus 0$. It may be written as $T_0(t) = (\exp \mathcal{A}_0 t) P_0$, $t \ge 0$. Every solution x to inclusion (4.1) for all $t \ge 0$ is represented in the form $x(t) = T_0(t) x_0$, $x_0 \in X_0 = \Phi(\mathcal{A})$.

Remark 4.3. Subspace X_0 under the conditions of assumption 4.2 is the phase space $\Phi(\mathcal{A})$ (subspace of initial data) for mild solutions [14].

Remark 4.4. Subspace X_∞, appearing under the conditions of assumption 4.1, according to theorem 4.1 does not contribute to the phase space $\Phi(\mathcal{A})$ by virtue of the nilpotency of operator $\mathcal{A}_\infty^{-1}$. However, if $\widetilde{\sigma}(\mathcal{A}) = \{\infty\}$, i. e. $\mathcal{A}^{-1} \in End\,X$ is a quasinilpotent operator, then it may appier that $\Phi(\mathcal{A}) = X$. An arbitrary operator $\mathcal{A} \in LO(X)$, which is a generator of a semigroup of class C_0 with $\widetilde{\sigma}(\mathcal{A}) = \{\infty\}$, may be such an example. In particular, the generator of a nilpotent semigroup of class C_0 (see [14]).

§ 5. Sectorial linear relations and analytic degenerate semigroups of operators

In this paragraph sectorial linear relations are defined and the construction of degenerate analytic semigroups of linear operators for them is carried out. In obtaining principal results the ergodic theorems are also used essentially.

Definition 5.1. The relation $\mathcal{A} \in LR(X)$ is called *sectorial* with angle $\theta \in (\pi/2, \pi)$, if for a certain $a \in \mathbb{R}$ the sector

$$\Omega = \Omega_{a,\theta} = \{\lambda \in C \mid |\arg(\lambda - a)| < \theta,\ \lambda \ne a\}$$

is contained in the resolvent set $\rho(\mathcal{A})$ of relation $\mathcal{A}$, and for every $\delta \in (0, \theta - \pi/2)$ the numbers $m \in \mathbb{N}$ and $M_\delta \ge 1$ exist such that

$$\sup_{\lambda \in \Omega_{a,\theta-\delta}} \left\| ((a - \lambda) R(\lambda, A))^m \right\| = M_\delta < \infty. \qquad (5.1)$$

Further in this paragraph it is assumed that holds

Assumption 5.1. Relation $\mathcal{A} \in LR(X)$ is sectorial.

To construct the analytic semigroup of operators, whose generator is a sectorial relation $\mathcal{A}$, we need

Lemma 5.1. *For the resolvent of a sectorial relation* $\mathcal{A} \in LR(X)$ *the constant* $C > 0$ *exists such that for all* $\delta \in (0, \theta - \pi/2)$ *the estimate*

$$\|R(\lambda, \mathcal{A})\| \le C(1 + |\lambda|)^{m-2},\ \lambda \in \Omega_{a,\theta-\delta} \qquad (5.2)$$

is valid.

Definition 5.2. Let $\mathcal{A}$ be a sectorial relation from $LR(X)$ with the angle θ.

Let us assume for $z \in \Omega_{0,\theta-\delta}$, where $\delta \in (0, \theta - \pi/2)$, that

$$T(z) = -\frac{1}{2\pi i} \int_\gamma e^{\lambda z} R(\lambda, \mathcal{A}) d\lambda. \qquad (5.3)$$

The union of three curves $\gamma_k(r, \varepsilon)$, $k = 1, 2, 3$ of the form $\gamma_1(r, \varepsilon) = \{(-\rho e^{-i(\theta-\varepsilon)} \mid -\infty \le -\rho \le -r\}$, $\gamma_2(r, \varepsilon) = \{r e^{i\alpha} \mid -(\theta - \varepsilon) \le \alpha \le \theta - \varepsilon\}$, $\gamma_3(r, \varepsilon) = \{\rho e^{i(\theta-\varepsilon)} \mid r \le \rho \le \infty\}$, where $\varepsilon = (\delta_0 - \delta)/2$, $\delta_0 = \theta - \pi/2$ and $r = 1/|z|$, can be used as the curve $\gamma = \gamma(r, \varepsilon)$ in definition 5.2.

The convergence of the integral from (5.3) in the uniform operator topology for all $z \in \Omega_{0,\delta}$, $\delta \in (0, \delta_0)$ follows from lemma 5.1.

Further it is assumed that $\delta_0 = \theta - \pi/2$.

Theorem 5.1. *Let* $\mathcal{A} \in LR(X)$ *be a sectorial relation (condition* (5.1) *from assumption* 5.1 *is fulfilled). Then equality* (5.4) *assigns the analytic in the sector* $\Omega_{0,\delta_0} \subset \rho(\mathcal{A})$ *and bounded for* $t > 0$ *semigroup of operators from algebra* $End\,X$.

The investigation of the semigroup $\{T(t); t > 0\}$ is conducted with the help of two operator-valued functions

$$\mathcal{A}(\lambda) = I - ((-\lambda R(\lambda, \mathcal{A}))^m),\ \lambda > 0, \qquad B(t) = I - T(t),\ t > 0. \qquad (5.4)$$

Further two arbitrary sequences (λ_n), (t_n) $\forall\, n \ge 1$ from $\mathbb{R}_+$ with the properties

$$\lambda_n \in \Omega \subset \rho(\mathcal{A}),\ \lim_{n\to\infty} \lambda_n = \infty,\ \lim_{n\to\infty} t_n = 0$$

and the corresponding bounded sequences of operators $\mathcal{A}_n = \mathcal{A}(\lambda_n)$, $B_n = B(t_n)$, $n \in \mathbb{R}$ from Banach algebra $End\,X$ are considered.

Lemma 5.2. *Under the conditions of assumption* 5.1 *for any fixed* $\lambda_0 \in \Omega_{0,\theta} \subset \rho(\mathcal{A})$ *the following properties are valid*:

1) $Im\,(I - A_n) \subset \overline{Im\,(R(\lambda_0, \mathcal{A}))^m}$,

$Im\,(I - B_n) \subset \overline{Im\,(R(\lambda_0, \mathcal{A}))^m}$;

2) $\lim_{n\to\infty} \mathcal{A}_n (R(\lambda_0, A))^m = 0$,

$\lim_{n\to\infty} B_n (R(\lambda_0, \mathcal{A}))^m = 0$.

Thus, according to the terminology from § 4 (definition 4.3) both sequences (A_n) and (B_n) are $\mathcal{J}$-sequences for the ideal $\mathcal{J} \subset \mathfrak{A}$ considered in § 4.

Theorem 5.2. *Under the conditions of assumption* 5.1 *the equalities*

$$Erg\,(X, (A_n)) = Erg\,(X, (B_n)) = \widetilde{X} = X_0 \oplus X_\infty \quad (5.5)$$

are valid, and all subspaces from the right parts in (5.5) *are closed. The equalities*

$$P_\infty x = \lim_{n\to\infty} A_n x = \lim_{n\to\infty} B_n x,\ x \in \widetilde{X}$$

define the bounded projector $P_\infty \in End\,\widetilde{X}$ *with the following properties*:

1) $\|P_\infty\| \le \min\{\varliminf_{n\to\infty} \|A_n\|, \varliminf_{n\to\infty} \|B_n\|\}$;

2) $Im\,P_\infty = X_\infty$, $Ker\,P_\infty = X_0$.

All statements of theorem 5.2 arise from [20], provided lemma 5.2 is used for its application.

Let us denote $P_0 = I - P_\infty$ and note that

$$P_0 x = \lim_{n\to\infty} (-\lambda_n R(\lambda_n, \mathcal{A}))^m x = \lim_{n\to\infty} T(t_n) x,\ x \in \widetilde{X}.$$

The indicated limits exist and are equal independent of the concrete form of the sequences $(\lambda_n), (t_n), n \in \mathbb{N}$ with the properties determined earlier. By the same token the representation of the subspaces X_0 and X_∞ from $\widetilde{X}$ is provided in the form

$$X_0 = \{x \in X \mid \lim_{t\to+0} T(t)x = x\},\ X_\infty = \bigcap_{t>0} Ker\,T(t). (5.6)$$

From this representation it follows that the restriction $\{T_0(t); t \ge 0\}$ of the semigroup $T(t)$ on the subspace X_0 is a semigroup strongly continuous in zero and analytic in the sector Ω_{0,δ_0}. The restriction $\{\widetilde{T}(z); z \in \Omega_{0,\delta_0}\}$ of the function $T : \Omega_{0,\delta_0} \to End\,X$ on the subspace $\widetilde{X}$ is analytic on the subspace X_0 too.

Let us denote by $\mathcal{O}_\infty = 0_\infty^{-1}$ the inverse relation to zero operator on X_∞.

Theorem 5.3. *Let* $\mathcal{A} \in LR(X) \setminus LO(X)$ *be a sectorial relation. Then* $\{\widetilde{T}(t); t \ge 0\} \subset End\,\widetilde{X}$ *is a semigroup of the operators analytic in sector* Ω_{0,δ_0}, *and it is a degenerate semigroup with the generator* $\widetilde{\mathcal{A}} = \mathcal{A}_0 \oplus \mathcal{O}_\infty \in LR(\widetilde{X})$, *where* $\mathcal{O}_\infty \in LR(X_\infty)$, $D(\mathcal{O}_\infty) = \{0\}$, $\mathcal{O}_\infty 0 = X_\infty$, *the operator* $\mathcal{A}_0 = \widetilde{A} \restriction X_0 \in LO(X_0)$ *is a generator of the semigroup of operators* $\{T_0(t); t \ge 0\}$ *strongly continuous and analytic in the sector* Ω_{0,δ_0}, *and* $\Phi(\mathcal{A}) \cap \widetilde{X} = X_0$. *An arbitrary solution to problem* (4.1)—(4.2) *with* $x_0 \in D(\mathcal{A}_0)$ *has the form* $x(t) = T_0(t)x_0$, $t \ge 0$.

If the vectors from the subspace $\mathcal{A}^m 0 \subset X$ *separate functionals from the subspace* $(\mathcal{A}^*)^m 0 \subset X^*$ *(in particular, if one of three conditions of theorem* 4.3 *is fulfilled), then* $\widetilde{X} = X$, *and then* $\Phi(\mathcal{A}) = X_0$.

§ 6. On the spectral theory of ordered pairs of linear operators

In this paragraph applications of the spectral theory of linear relations to the spectral theory of ordered pairs (G, F) of linear closed operators $F : D(F) \subset X \to Y$, $G : D(G) \subset X \to Y$, mapping from complex Banach space X to complex Banach space Y, are obtained.

Domains $D(F), D(G)$ will be considered to satisfy one of the following conditions:

(i) $D(F) = X$, $D(G) \ne X$;

(ii) $D(F) \ne X$, $D(G) = X$;

(iii) $D(F) = X$, $D(G) = X$.

The subspace $D(F) \cap D(G)$ is denoted by $\mathcal{D} = \mathcal{D}(G, F)$ and is called *the domain* of the ordered pair of operators (G, F).

Definition 6.1. To *the resolvent set* $\rho(G,F)$ of the ordered pair of operators (G,F) we refer all numbers $\lambda \ne 0$ from $\mathbb{C}$, for which $G - \lambda F : \mathcal{D} \subset X \to Y$ is a continuously reversible operator and, besides, point $\lambda = 0$, if $G : \mathcal{D} \to Y$ is a continuously reversible operator and $D(F) = X$. The set $\sigma(G, F) = \mathbb{C} \setminus \rho(G, F)$ is called *the spectrum* of this pair.

The operator-valued function

$$R(\,.\,; G, F) : \rho(G, F) \subset \mathbb{C} \to Hom\,(Y, X),$$

$$R(\lambda; G, F) = (G - \lambda F)^{-1},\ \lambda \in \rho(G, F)$$

is called *the resolvent* of the ordered pair (G,F). It is defined on the open set $\rho(G, F)$ and it is analytic there.

Remark 6.1. If the condition $0 \in \rho(G, F)$ is understood only formally as the continuous reversibility of the operator G, then point 0 is isolated in $\rho(G,F)$ in case (ii), and so the set $\rho(G,F)$ is not open. Also a variety of other problems arises.

Here the introduction of the notion of the extended spectrum of the ordered pair provides an especially useful guide as well as in the case of linear relations.

Definition 6.2. The subspace $\widetilde{\sigma}(G,F)$ from $\widetilde{\mathbb{C}}$, coinciding with $\sigma(G,F)$, when the function $R(.;G,F)$ admits the analytic extension at point ∞ provided $\lim\limits_{\|\lambda\|\to\infty} R(\lambda;G,F)=0$, and $\widetilde{\sigma}(G,F)=\sigma(G,F)\cup\{\infty\}$ in the opposite case is called *the extended spectrum* of the ordered pair (G,F). The set $\widetilde{\rho}(G,F)=\widetilde{\mathbb{C}}\setminus\widetilde{\sigma}(G,F)$ is called *the extended resolvent set* of the pair (G,F).

When the reducing of problem (1.3)—(1.2) to problem (4.1)—(4.2) takes place, two relations $\mathcal{A}_l=F^{-1}G\subset X\times X$, $\mathcal{A}_r=GF^{-1}\subset Y\times Y$ arise naturally and are called *the left and the right relation* for the ordered pair (G,F), respectively.

From the definitions it follows that for the left $\mathcal{A}_l$ and the right $\mathcal{A}_r$ relation, constructed on the ordered pair (G,F), the following representations are valid:

$$D(\mathcal{A}_l)=G^{-1}(Im\,F),\ Im\,A_l=F^{-1}(Im\,G), \quad (6.1)$$

$$D(\mathcal{A}_r)=F(D(G)),\ Im\,\mathcal{A}_r=G(D(F)), \quad (6.2)$$

$$R(\lambda,\mathcal{A}_l)=(G-\lambda F)^{-1}F,\ \lambda\in\rho(G,F), \quad (6.3)$$

$$R(\lambda,\mathcal{A}_r)=F(G-\lambda F)^{-1},\ \lambda\in\rho(G,F). \quad (6.4)$$

Formula (6.3) is true only if $D(G),D(F)$ obey one of the conditions (i) or (iii).

If condition (ii) is fulfilled, then formula (6.3) is incorrect, since $D(F)\neq X$. In this case one may apply formula (2.4) to $G^{-1}F$ for $0\neq\lambda\in\rho(G,F)$, as a result we obtain the correlation

$$\begin{aligned} R(\lambda,\mathcal{A}_l) &= -\lambda^{-1}I-\lambda^{-2}(\mathcal{A}_l^{-1}-\lambda^{-1}I)^{-1} = \\ &= -\lambda^{-1}I-\lambda^{-2}(G^{-1}F-\lambda^{-1}I)^{-1} = \\ &= -\lambda^{-1}(I-(G-\lambda F)^{-1}G). \end{aligned} \quad (6.5)$$

From representations (6.3)—(6.5) it follows that if $\infty\in\widetilde{\rho}(G,F)$, then $\infty\notin\widetilde{\sigma}(\mathcal{A}_l)\cup\widetilde{\sigma}(\mathcal{A}_r)$, and so $A_l\in End\,X$, $A_r\in End\,Y$. Thus, we obtain the following

Lemma 6.1. *The inclusions*

$$\begin{aligned} \widetilde{\rho}(G,F) &\subset \widetilde{\rho}(\mathcal{A}_l)\cap\widetilde{\rho}(\mathcal{A}_r), \\ \widetilde{\sigma}(G,F) &\supset \widetilde{\sigma}(\mathcal{A}_l)\cup\widetilde{\sigma}(\mathcal{A}_r) \end{aligned} \quad (6.6)$$

take place.

The resolvents of relations $\mathcal{A}_l$ and $\mathcal{A}_r$ are called *the left and the right resolvent* of the ordered pair of operators (G,F) and they are denoted by symbols $R_l(.;G,F)$ and $R_r(.;G,F)$, respectively. The values of these functions, by definition, are in algebras $End\,X$ and $End\,Y$, respectively.

Further it is supposed that the following condition of *nonsingularity* of the pair (G,F)) is fulfilled.

Assumption 6.1. For the ordered pair (G,F) the set $\rho(G,F)$ is not empty.

Remark 6.2. In monograph [2] the conditions $D(F)\subset D(G)\subset X=Y$ are fulfilled. In papers [11, 12] operators F,G with the properties $F\in Hom\,(X,Y)$, $\overline{D(G)}=X$ were considered. The most general case is studied in [23], where $D(F),D(G)$ cannot coincide with X, simultaneously. However, the investigation was carried out under the assumption that $\mathcal{D}=D(F)\cap D(G)\neq\{0\}$ and, moreover, under our assumption 6.1. It allows us to introduce the norm $\|x\|_*=\|(G-\lambda_0F)x\|$, $x\in\mathcal{D}$, where λ_0 is a certain number from $\rho(G,F)$, on $\mathcal{D}$. With respect to this norm $\mathcal{D}$ is a Banach space isomorphic to Y. Considering $\mathcal{D}$ instead of X, one may regard that condition (iii) is fulfilled and take one of the subspaces $D(F),D(G)$ with the corresponding graph norm as a Banach space containing $\mathcal{D}$.

Let us select, along with conditions (i)—(iii), the following conditions:

(iv) $D(F)\subset D(G)\neq X$;

(v) $D(G)\subset D(F)\neq X$.

Then, if condition (iv) is fulfilled, then on $D(G)$ a graph norm of operator $G:\|x\|_*=\|x\|+\|Gx\|$, $x\in D(G)$ is introduced, and considering $D(G)$ instead of X, one may regard that condition (ii) is fulfilled. If condition (v) is fulfilled, then on $D(F)$ a graph norm of operator F is introduced, and we have the conditions, when (i) is fulfilled.

Theorem 6.1. *For the ordered pair of operators (G,F) the following properties take place:*

1) $\widetilde{\sigma}(G,F)=\widetilde{\sigma}(\mathcal{A}_l)$;
2) $\widetilde{\sigma}(G,F)\setminus\{0,\infty\}=\widetilde{\sigma}(\mathcal{A}_r)\setminus\{0,\infty\}$;
3) $\widetilde{\sigma}(G,F)=\widetilde{\sigma}(\mathcal{A}_l)=\widetilde{\sigma}(\mathcal{A}_r)$, *if* $\mathcal{D}=X$;
4) $0\in\rho(\mathcal{A}_l)\Longleftrightarrow G^{-1}F\in End\,X$;
5) $0\in\rho(\mathcal{A}_r)\Longleftrightarrow FG^{-1}\in End\,Y$;
6) $0\in\rho(\mathcal{A}_l)\cap\rho(\mathcal{A}_r)\Longleftrightarrow 0\in\rho(G,F)$;
7) $\infty\in\widetilde{\rho}(\mathcal{A}_l)\Longleftrightarrow\mathcal{A}_l=F^{-1}G\in End\,X$;
8) $\infty\in\widetilde{\rho}(\mathcal{A}_r)\Longleftrightarrow\mathcal{A}_r=GF^{-1}\in End\,Y$;
9) $\infty\in\widetilde{\rho}(\mathcal{A}_l)\cap\widetilde{\rho}(\mathcal{A}_r)\Longleftrightarrow\infty\in\widetilde{\rho}(G,F)$.

Corollary 6.1. *If points $0, \infty$ are contained in $\widetilde{\rho}(G,F)$, simultaneously, then $\mathcal{D} = X$ and operators $G, F \in Hom(X,Y)$ are continuously reversible.*

Corollary 6.2. *The ordered pair (G,F) possesses the following properties:*

a) $\widetilde{\sigma}(G,F) = \{0\} \Longleftrightarrow$ *operator $F : D \to Y$ is continuously reversible, $D(G) = X$ and operators $\mathcal{A}_l = F^{-1}G \in End\, X$, $\mathcal{A}_r = GF^{-1} \in End\, Y$ are quasinilpotent;*

b) $\widetilde{\sigma}(G,F) = \{\infty\} \Longleftrightarrow$ *operator $G : \mathcal{D} \to Y$ is continuously reversible, $D(F) = X$ and operators $\mathcal{A}_l^{-1} = G^{-1}F \in End\, X$, $\mathcal{A}_r^{-1} = FG^{-1} \in End\, Y$ are quasinilpotent.*

Let us denote $S_0(\delta) = \{0 \neq \lambda \in \mathbb{C} : |\lambda| < \delta,\ \delta > 0\}$.

Remark 6.3. The following two conditions are equivalent:

a) $D(F) = X$ and $G : D(G) \subset X \to Y$ is a continuously reversible operator;

b) $\delta > 0$ exists such that $S_0(\delta) \subset \rho(G,F)$ and

$$\sup_{\lambda \in S_0(\delta)} \left\| (G - \lambda F)^{-1} \right\| < \infty.$$

Note that the sets $\widetilde{\sigma}(G,F) = \widetilde{\sigma}(\mathcal{A}_l)$ and $\widetilde{\sigma}(\mathcal{A}_r)$ can be distinguished.

Example 6.1. Let $\mathcal{H}$ be a complex Hilbert space and let $T : \mathcal{H} \to \mathcal{H}$ be an irreversible isometry. Then operator T^* has a nonzero kernel, moreover, $TT^* \neq I$, $T^*T = I$. Let us consider at first the ordered pair (G,F), where $G = T^* \in End\,\mathcal{H}$, $F = T^{-1} : Im\, T \subset \mathcal{H} \to \mathcal{H}$. Then $0 \in \sigma(G,F)$. At the same time $A_l = TT^* \neq I$ and $A_r = T^*T = I$, consequently, $0 \in \sigma(\mathcal{A}_l)$, but $0 \notin \sigma(\mathcal{A}_r) = \{1\}$. Further, let us consider the ordered pair (F,G). Since the left and the right relation coincides with $\mathcal{A}_l^{-1}$ and $\mathcal{A}_r^{-1}$, respectively, then according to theorems 2.4 and 6.1 we obtain $\infty \in \widetilde{\sigma}(F,G) = \widetilde{\sigma}(\mathcal{A}_l^{-1})$, but $\infty \notin \widetilde{\sigma}(\mathcal{A}_r^{-1})$. At last, both the cases are united by the consideration of a suitable pair of operators in $\mathcal{H} \times \mathcal{H}$.

Theorem 6.2. *The extended spectrums of the ordered pairs (G,F) and (F,G) are connected by the correlation*

$$\widetilde{\sigma}(F,G) = \{1/\lambda \mid \lambda \in \widetilde{\sigma}(G,F)\}. \qquad (6.7)$$

Definition 6.3. The ordered pair of subspaces (X_1, Y_1), where $X_1 \subset X$, $Y_1 \subset Y$, is called *invariant* for the pair (G,F), if $GX_1 \subset Y_1$ and $FX_1 \subset Y_1$.

Definition 6.4. Let

$$X = X_0 \oplus X_1,\ Y = Y_0 \oplus Y_1 \qquad (6.8)$$

be direct sums of closed subspaces provided (X_0, Y_0), (X_1, Y_1) are invariant pairs of subspaces for (G,F). Let $G_i, F_i : \mathcal{D}(G,F) \cap X_i = \mathcal{D}_i \to Y_i$, $i = 0,1$ be restrictions of operators G, F on X_i, $i = 0,1$. Then we shall use the notation

$$(G,F) = (G_0,F_0) \oplus (G_1,F_1) \qquad (6.9)$$

and we shall say that the ordered pair of operators (G,F) assumes the representation (6.12) according to expansions (6.11) of spaces, and it is *a direct sum* of pairs (G_0,F_0) and (G_1,F_1).

Theorem 6.3. *Let the extended spectrum $\widetilde{\sigma}(G,F)$ of the ordered pair (G,F) be represented in the form*

$$\widetilde{\sigma}(G,F) = \sigma_0 \cup \sigma_1, \qquad (6.10)$$

where set σ_0 is compact, set σ_1 is closed and $\sigma_0 \cap \sigma_1 = \varnothing$. Then the pairs of subspaces (X_0,Y_0), (X_1,Y_1) invariant for (G,F) exist such that expansions (6.8), (6.9) take place and, besides,

1) *projectors $P_i \in End\, X$, $Q_i \in End\, Y$, $i = 0,1$, realizing expansions (6.8) (i. e. $Im\, P_i = X_i$, $Im\, Q_i = Y_i$, $i = 0,1$), are defined by the formulae*

$$P_0 = -\frac{1}{2\pi i}\int_{\gamma_0} R(\lambda, \mathcal{A}_l)\, d\lambda,$$
$$Q_0 = -\frac{1}{2\pi i}\int_{\gamma_0} R(\lambda, \mathcal{A}_r)\, d\lambda, \qquad (6.11)$$

$$P_1 = I - P_0,\ Q_1 = I - Q_0,\ i = 0,1, \qquad (6.12)$$

where γ_0 is a closed Jordan circle (or a finite number of such circles), placed in $\rho(\mathcal{A})$ so that σ_0 lies inside it, and σ_1 lies outside of it;

2) $\widetilde{\sigma}(G_0,F_0) = \sigma(G_0,F_0) = \sigma_0$, $\widetilde{\sigma}(G_1,F_1) = \sigma_1$;

3) *$D(G_0) = X_0$, $F_0 : D(F_0) = D(F) \cap X_0 \subset X_0 \to Y_0$ is a continuous reversible operator, and $\mathcal{A}_l^{(0)} = F_0^{-1}G_0 \in End\, X_0$, $\mathcal{A}_r^{(0)} = G_0F_0^{-1} \in End\, Y_0$;*

4) *the left $R_l(.;G_0,F_0) = R(.,\mathcal{A}_l^{(0)})$ and the right $R_r(.;G_0,F_0) = R(.,\mathcal{A}_r^{(0)})$ resolvents of pair (G_0,F_0) are similar, and $\widetilde{\sigma}(\mathcal{A}_l^{(0)}) = \sigma(\mathcal{A}_l^{(0)}) = \sigma(\mathcal{A}_r^{(0)}) = \sigma(\mathcal{A}_r^{(0)}) = \widetilde{\sigma}(G_0,F_0)$;*

5) *$D(F_1) = X_1$, $G_1 : D(G_1) = D(G) \cap X_1 \subset X_1 \to Y_1$ is a continuous reversible operator, if $0 \notin \sigma_1$, and then $R_l(0;G_1,F_1) = G_1^{-1}F_1 \in End\, X_1$, $R_r(0;G_1,F_1) = F_1G_1^{-1} \in End\, Y_1$, and $R_l(.;G_1,F_1) = 0$, $R_r(.;G_1,F_1) = 0$, if $X_1 = \mathcal{A}_l\, 0$.*

Remark 6.4. If subspace $D(F)$ or $D(G)$ with the corresponding graph norm is chosen as space X according to remark 6.2, then in theorem 6.3 instead of (6.11) the expansion of subspace $D(F)$ or $D(G)$ is realized.

Theorem 6.3 was unknown to us in such a general formulation. Many of its statements were obtained earlier under specific conditions on the domain $D(G,F)$ of pair (G,F). Thus, in paper [24] A. G. Rutkas formulated without proof a portion of statements 1)—3) of theorem 6.3 in the case, when $D(G,F)=X$. The case $\sigma_0=\{0\}$, $D(G,F)=D(F)$ was explicitly considered in [25] and further in [26]. In paper [27] the result was obtained by V. V. Ditkin, which is contained in statements 1)—3) of theorem 6.3 provided $D(F)\subset D(G)\subset X$ and $X=Y$ (see remarks 6.2, 6.4). The most general result for the pair (G,F) of the closed operators was considered by N. I. Radbel in [23] (see remark 6.4). However, in formulae of the form (6.15) the resolvent of relation $\mathcal{A}_l$ was replaced by the right part of formula (6.3), but it is possible only in the case $D(F)=X$.

In the papers, mentioned above, possibilities and advantages, connected with the invoking of a extended spectrum of the pair, were not properly used.

In example 6.1 it was noted, that the sets $\widetilde{\sigma}(G,F)$ and $\widetilde{\sigma}(\mathcal{A}_r)$ can be distinguished by the presence or absence of points $0,\infty$. Their distinction is characterized more exactly by

Theorem 6.4. *The following two statements take place:*

1) $0\in\sigma(G,F)\setminus\sigma(A_r)\Longleftrightarrow Ker\,G\neq\{0\}$,

$Y=Im\,G$, $\overline{D(F)}=D(F)$ *and* $X=Ker\,G\oplus D(F)$;

2) $\infty\in\widetilde{\sigma}(G,F)\setminus\widetilde{\sigma}(\mathcal{A}_r)\Longleftrightarrow KerF\neq\{0\}$,

$Y=Im\,F$, $\overline{D(G)}=D(G)$ *and* $X=Ker\,F\oplus D(G)$.

Theorem 6.5. *Let* $B_1\in Hom(X,Y)$, $B_2\in Hom(Y,X)$, *and also* $KerB_2=\{0\}$. *Then* $\sigma(B_1B_2)\setminus\{0\}=\sigma(B_2B_1)\setminus\{0\}$. *Moreover,* $0\in\sigma(B_2B_1)\setminus\sigma(B_1B_2)\Longleftrightarrow Ker\,B_1\neq\{0\}$, $Im\,B_1=Y$, $\overline{Im\,B_2}=Im\,B_2$ *and* $X=Ker\,B_1\oplus Im\,B_2$.

Note that the statement of theorem 6.5 for the elements of Banach algebras is contained in many monographs (see, for example, [16, ch. 1, § 1]). The detailed analysis of the spectral properties, corresponding to the second part of theorem 6.5, was not carried out there.

Let us formulate the results, which are closely connected with the results from § 3 and are their direct corollary.

With the help of the left $\mathcal{A}_l$ and the right $\mathcal{A}_r$ relation of pair (G,F) let us introduce into consideration sequences of linear subspaces

$$\mathcal{X}_k=\mathcal{A}_l^k\,0,\ \mathcal{X}^{(k)}=D(\mathcal{A}_l^k),\ \mathcal{Y}_k=\mathcal{A}_r^k\,0,\ \mathcal{Y}^{(k)}=D(\mathcal{A}_r^k),\ k\in\mathbb{N}. \tag{6.13}$$

From properties 1)—3) of the relations, given at the beginning of this paragraph, we obtain the representation of subspaces in terms of images and preimages of operators F and G:

$$\mathcal{X}_0=\{0\},\ \mathcal{X}_1=Ker\,F,\ldots,$$
$$\mathcal{X}_n=F^{-1}(GX_{n-1}),\ldots,\ n\in\mathbb{N},$$
$$\mathcal{X}^{(0)}=\mathcal{X},\ \mathcal{X}^{(1)}=G^{-1}(Im\,F),\ldots,$$
$$\mathcal{X}^{(n)}=G^{-1}(F\mathcal{X}^{(n-1)}),\ldots,\ n\in\mathbb{N},$$
$$\mathcal{Y}_0=\{0\},\ \mathcal{Y}_1=G(Ker\,F),\ldots,$$
$$\mathcal{Y}_n=G(F^{-1}\mathcal{Y}_{n-1}),\ldots,\ n\in\mathbb{N},$$
$$\mathcal{Y}^{(0)}=\mathcal{Y},\ \mathcal{Y}^{(1)}=F(D(G)),\ldots,$$
$$\mathcal{Y}^{(n)}=F(G^{-1}\mathcal{Y}^{(n-1)}),\ldots,\ n\in\mathbb{N}.$$

It is clear that the pairs of subspaces $(\mathcal{X}_n,\mathcal{Y}_n)$, $(\mathcal{X}^{(n)},\mathcal{Y}^{(n)})$ are invariant for the pair of operators (G,F).

Under the conditions of the following theorem $m\in\mathbb{N}$, $m\geq 2$, and inclusions are strict.

Theorem 6.6. *If* $Ker\,F\neq\{0\}$, *then for the ordered pair* (G,F) *the following conditions are equivalent:*

1) *point* ∞ *is a pole of the resolvent of the left relation* $\mathcal{A}_l$ *of pair* (G,F) *of the order* $m-1$ *for* $m\geq 2,\infty$ *is its removable singularity for* $m=1$;

2) *invariant pairs of subspaces* $(X_0,Y_0),(X_1,Y_1)$ *exist, for which representations (6.11), (6.12) are valid, and*

a) $\widetilde{\sigma}(G_0,F_0)=\sigma(G,F)$, $\widetilde{\sigma}(G_1,F_1)=\{\infty\}$;

b) $D(G_0)=X_0$, $F_0^{-1}\in Hom(Y_0,X_0)$, $D(F_1)=X_1$, $G_1^{-1}\in Hom(Y_1,X_1)$;

c) $(G_1^{-1}F_1)^{m-1}\neq 0$, $(G_1^{-1}F_1)^m=0$;

3) *the stability of subspaces takes place*: $\mathcal{X}_{m-1}\subset\mathcal{X}_m=\mathcal{X}_{m+1}$, $\mathcal{X}^{(m-1)}\supset\mathcal{X}^m=\mathcal{X}^{(m+1)}$.

To obtain the analogs of theorems 4.3, 4.4, 5.3 in terms of operator pairs one may take subspaces $(\mathcal{X}^*)_m=(\mathcal{A}_l^*)^m\,0\subset\mathcal{X}^*$, which are described by analogy with (6.13), as

$$\mathcal{X}_0^*=\{0\},\ \mathcal{X}_1^*=G^*(Ker\,F^*),\ldots,$$
$$\mathcal{X}_n^*=G^*((F^*)^{-1}\mathcal{X}_{n-1}^*),\ldots,\ n\in\mathbb{N}.$$

REFERENCES

1. *Cross R.* Multivalued linear operators. New York: M. Dekker, 1998.

2. *Favini A., Yagi A.* Degenerate differential equations in Banach spaces. Pure and Applied Mathematics. A Series of Monographs and Textbooks / 215. New York: M. Dekker, 1998.

3. *Ritsner V. S.* Theory of linear relations // Deposited in VINITI. 1982. № 846—82. (in Russian)

4. *Hille E., Phillips R.* Functional analysis and semigroups. M.: IL, 1962. (in Russian)

5. *Dunford N., Schwartz J.T.* Linear operators. P. I: General theory. M.: IL, 1962. (in Russian)

6 *Neumann J. von.* Über adjungierte Functionaloperatoren // Ann. of Math. (1932), № 2, 294—310.

7. *Arendt W.* Approximation of degenerate semigroups // Tübinger Berichte zur Funktionalanalysis. Heft 9 (1999/2000), 33—46.

8. Functional analysis, ser. SMB, edit. Krein S. G. M.: Nauka, 1972. (in Russian)

9. *Baskakov A. G., Chernyshov K. I.* Ordered pairs of operators and semigroups // Izv. RAEN. MMMIU, (1998), № 3, 39—69. (in Russian)

10. *Baskakov A. G., Chernyshov K. I.* Construction of the phase space and solutions of linear equations unsolved with respect to the derivative // Dokl. RAN, (2000), № 3, 295—298. (in Russian)

11. *Sviridyuk G. A., Fedorov V. E.* On units of analytic semigroups of operators with kernels // Sib. mat. zh., (1998), № 3, 604—616. (in Russian)

12. *Fedorov V. E.* Degenerate strongly continuous semigroups of operators // Algebra i analiz, (2000), № 3, 173—200. (in Russian)

13. *Melnikova I. V., Gladchenko A. V.* Correctness of Cauchy problem for inclusions in Banach spaces // Dokl. RAN, (1998), № 6, 736—739. (in Russian)

14. *Engel K.-J., Nagel R.* One-parameter Semigroups for Linear Evolution Equations. Gradute Texts in Mathematics. Springer-Verlag, 2000.

15. *Räbiger F., Wolff M. P. H.* Spectral and asymptotic properties of resolvent-dominated operators // Tübinger Berichte zur Funktionalanalysis. Heft 7 (1997/1998), 217—235.

16. *Bourbaki N.* Spectral theory. M.: Mir, 1972. (in Russian)

17. *Favini A., Yagi A.* Multivalued linear operators and degenerate evolution equations // Ann. Math. pur. et appl. CLXIII (1993), 353—384.

18. *Pietsch A.* Operator ideals. M.: Mir, 1982. (in Russian)

19. *Baskakov A.G., Chernyshov K.I.* On the spectrum compactness conditions of ordered pairs of linear operators // Izv. RAEN. MMMIU, (1999), № 3, 5—24. (in Russian)

20. *Baskakov A.G.* Operator ergodic theorems and the complementability of subspaces of Banach spaces // Izv. vuzov, ser. math., (1988), № 11 (318), 3—11. (in Russian)

21. *Krein S.G.* Linear differential equations in Banach space. M.: Nauka, 1967. (in Russian)

22. *Henry D.* Geometric theory of semilinear parabolic equations. M.: Mir, 1985. (in Russian)

23. *Radbel N.I.* Ordered pairs of linear operators and the Cauchy problem for equation $A\dot{x}(t) + Bx(t) = 0$ in a Banach space. Disser. kand. phys.-math. nauk. Donetsk: IPMM, 1984. (in Russian)

24. *Rutkas A.G.* Cauchy problem for equation $A\dot{x}(t) + Bx(t) = f(t)$ // Differen. uravneniya, (1975), № 11, 1996—2010. (in Russian)

25. *Chernyshov K.I.* On operator differential equations unsolved with respect to the derivative. Disser. kand. phys.-math. nauk. Kiev: IM AN USSR, 1979. (in Russian)

26. *Krein S.G., Chernyshov K.I.* Singularly perturbed differential equations in Banach space // IX International conf. on nonlinear oscillations. Kiev: Nauk. dumka. (1984), 193—197. (in Russian)

27. *Ditkin V.V.* On certain spectral properties of the bundle of linear bounded operators // Matem. zametki, (1982), № 1, 75—79. (in Russian)

28. *Sviridyuk G.A.* To the general theory of semigroups of operators // Uspekhi Mat. Nauk. (1994), № 4, 47—74. (in Russian)

29. *Baskakov A.G., Chernyshov K.I.* Ergodic subspaces and analytic semigroups // Spectral and evolutionary problems. Simferopol. (2001), 136—143.