РАЗДЕЛ МАТЕМАТИКА

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INVESTIGATIONS OF PROPERTIES OF ATTRACTORS FOR A REGULARIZED MODEL OF THE MOTION OF A NONLINEAR-VISCOUS FLUID

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1. Introduction: nonlinear approximation

It is known that many real fluids are characterized by a nonlinear relation between the shearing stress and shear speed. The phenomenological description of the flow of such fluids has been the object of consideration of mechanicians during the last sixty years. The survey of suggested models and of their rheological properties can be found, for example, in the known monograph G. Astarita and G. Marrucci [3] and in the fundamental work C. Truesdell and W. Noll [15].

The mathematical investigation of Reiner—Rivlin models was apparently for the first time carried out by W. G. Litvinov [11]. Here, as well as for Navier-Stokes equations, the problem of proving the solvability of the Cauchy problem on an arbitrary time interval arises in a strong form. To overcome the difficulties arising here various ε-approximations of appropriate equations were suggested which goes back to the known paper R. Temam [14].

For the mathematical investigation of models of nonlinear-viscous fluids from [11] a D_ε-approximation was used for inertia terms

$$D_\varepsilon(v) = \left(\frac{v}{1+\varepsilon |v|^2} \operatorname{grad} \right) v, \qquad \varepsilon \ge 0, \quad |v|^2 = \sum_{i=1}^{n} v_i^2, \; n = 2,3, \tag{1}$$

suggested by P. E. Sobolevskii in [12] was applied, where it was proved that the equations of the motion of nonlinear-viscous fluid, regularized in such way, have for $\varepsilon > 0$ unique strong solution on any finite time segment, both in the cases of two and three space variables. The last in turn, means that we have reason to study the minimal global attractor of the given system, i.e. set $\mathfrak{M}$ is equal intersection the sets $V_t(X)$, $t \ge 0$, where X is a phase space and V_t is some semigroup with operators for the evolutionary problem. We will call minimal global B-attractor for a semigroup a minimal nonempty closed set in X, that attracting any bounded subset B in X. But, in our sense attractor is some subset in phase space on that a semigroup V_t expand to a group V_t, $t \in \mathbb{R}$. The estimates proved in what follows show that the quantity ε, which we will call later on the inertia parameter, plays for the attractor the same role as the viscosity of the fluid. In other words the fact that the fluid "forgets" its initial data is stipulated not only by dissipation of energy but also by relaxation of the inertia forces. We explain it more in detail: $D\varepsilon(v)$ can be represented in the form:

$$D_\varepsilon(v) = \frac{1}{2\varepsilon} \operatorname{grad}[\ln(1+\varepsilon |v|^2)] - \frac{v \times \operatorname{rot} v}{1+\varepsilon |v|^2}, \tag{1'}$$

therefore $(D_\varepsilon(v), v)_{L_2(\Omega)} = 0$ $(\varepsilon \ge 0)$ for the boundary conditions of "adhesion" $v|_{\partial\Omega} = 0$ and consequently $D_\varepsilon(v)$ does not contribute to the dissipation of energy.

The authors realize that the estimates obtained in what follows are only unilateral (upper ones). Therefore our analysis of the influence of the physical parameters of the system on the geometric characteristics of its attractor has a nature of plausible reasoning. We do not know yet the way of obtaining analogous lower bounds.

Later on we follow the terminology of the attractor theory for Navier-Stokes and close evolutionary equations, which was developed by O. A. Ladyzhenskaya (see, for instance, [9]). Problems of the attractor theory for nonregularized ($\varepsilon = 0$) equations of nonlinear visco-elastic fluid with an exponential kernel were studied by A. P. Oskolkov and his collaborators in the case of two space variables [4] and also for Kelvin-Voigt fluid in the case of three space variables [5].

Let Ω be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega \in C^2$. By $H(\Omega)$ we denote the closure in the $L_2(\Omega)$-norm of the space of compactly supported and solenoindal in Ω vector-functions; $\overset{\circ}{D}(\Omega)$ is the Sobolev space $\overset{\circ}{W}{}_2^1(\Omega)$ with Dirichlet norm; A is the Friedrichs extension of Stokes operator corresponding to the stationary problem for the Navier-Stokes equations with the boundary conditions of "adhesion"; P_N is the orthoprojector on the subspace of $H(\Omega)$ which corresponds to the first N eigenfunctions of operator A; $Q_N = I - P_N$; λ_N is the N-th eigenvalue of operator A. We recall that $\lambda_N = O(N^{2/3})$ in the case of three space variables.

It will be convenient for us to write the Laplace operator acting on the vector field u defined on $\bar{\Omega}$, in the following form

$$\Delta u \equiv \mathrm{Div\,Grad}\ u \equiv \mathrm{grad\,div}\, u - \mathrm{rot\,rot\,u},$$

here Grad u denotes the covariant derivative (or the local affinor) of vector u, the symbol Div denotes the covariant derivative of a tensor, in our case of the tensor Grad u. In other words we accept the usual, in mechanics, definitions the Laplacian as a second order defferential scalar invariant (see, for instance, [6], 16.10-7).

Consider the following autonomous system of equations

$$\frac{\partial v}{\partial t} + D_\varepsilon(v) + \nu\,\mathrm{rot\,rot}\,v + B(v) + \mathrm{grad}\,p = f(x),\ \mathrm{div}\,v = 0$$
$$(t \geq \tau, x \in \Omega),\ \ \nu = const. > 0,\ \ \varepsilon > 0,\ \ v = \{v_1, v_2, v_3\}. \tag{2}$$

with the boundary condition of "adhesion": $v(t,x) = 0$ ($t \geq \tau$, $x \in \partial\Omega$; with the initial condition: $v(\tau,x) = v^\tau(x)$ ($x \in \bar{\Omega}$, $\bar{\Omega} = \Omega \cup \partial\Omega$); with the condition of the fluid, as a whole, does not produce any mechanical work with the environment: $(p,1)_{L_2(\Omega)} = \int_\Omega p(t,x)dx = 0$

Here we denote by $B(v) = -\mathrm{Div}\,[2\mu(I_2)\mathcal{E}]$, $\mathcal{E}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$ is the tensor of deformation velocities or the Cauchy tensor; $I_2^2 = \mathcal{E} : \mathcal{E} = \sum_{i,j=1}^{3} \mathcal{E}_{ij}^2$ is the second invariant of tensor $\mathcal{E}$. The symbol ":" denotes the convolution of second rank tensors over two indices, what corresponds to the scalar product of matrices $A : \mathrm{B} \equiv (A,B) \equiv sp(B^*A)$.

Let T be an arbitrary fixed number, $T > \tau$, $Q = [\tau, \mathrm{T}] \times \Omega$. Under a strong solution of problem (2) we mean a pair $\{v(t,x),\ p(t,x)\} \in \{W_2^{1,2}(Q), W_2^{0,1}(Q)\}$ satisfying all conditions and equations (2), with all addends of the equations belonging to $L_2(Q)$.

The following conditions are necessary for the existence of strong solutions of problem (2):

$$f(x) \in L_2(\Omega); v^\tau(x) \in \overset{\circ}{D}(\Omega) \cap H(\Omega). \tag{3}$$

It was proved in [12] that the sufficient conditions of the unique solvability, in the sense indicated above, for problem (2) are the following:

$$\begin{aligned} &(i)\ 0 \leq \mu(s) \leq M_1 < \infty; \mu(s) \text{ is a continuously differentiable function and if } \mu'(s) < 0, \\ &\quad \text{then } -s\mu'(s) \leq \mu(s); \\ &(ii)\ s|\mu'(s)| \leq M_2 < \infty. \end{aligned} \tag{4}$$

Later on we will assume conditions (3) and (4) fulfilled.

As the phase space of the system we will consider the space $X = \overset{\circ}{D}(\Omega) \cap H(\Omega)$ with $L_2(\Omega)$-norm.

2. An absorbing set

By $V_{t,\tau}$ we denote the semigroup of nonlinear operators solving the problem (2) with respect to the first component: $V_{t,\tau}[v_\tau(x)] = v(t,x)$, moreover $f(x)$ is assumed to be an arbitrary fixed function from $L_2(\Omega)$ (without loss of generality we may assume, that $f(x) \in H(\Omega)$).

We will call the subset F of the phase space X absorbing for the family $V_{t,\tau}$ if all trajectories of the initial boundary value problem (2) get into and remain in F in finite time intervals (see, for example [9]).

Theorem 1. *Any ball* $O_R = \{u(x) : u(x) \in X,$ $\|u\|_{L_2(\Omega)} \leq R\}$ *of radius* $R > R_0 = \frac{\|f\|_{L_2(\Omega)}}{\lambda_1(\nu + C_2)}$, *where* $C_2 = 2\min_{s\geq 0}|\mu(s)|$, *is an absorbing set for the family* $V_{t,\tau}$.

Proof. We fix an arbitrary function $v^\tau(x) \in X$ and consider the corresponding trajectory $v(t,x)$ for $t \geq \tau$. Multiplying scalarly in $L_2(\Omega)$ the first equation of (2) by $v(t,x)$, we get for almost all $t \geq \tau$ the following equality

$$\frac{1}{2}\frac{d}{dt}\|v(t,\cdot)\|^2_{L_2(\Omega)} + \nu\|v(t,\cdot)\|^2_{\overset{\circ}{D}(\Omega)} + (B(v),v)_{L_2(\Omega)} = = (f,v)_{L_2(\Omega)}. \tag{5}$$

From condition (4) (i) we obtain the following lower bound of the nonlinear term on the left-hand side of (5):

$$(B(v),v)_{L_2(\Omega)} = -2\int_\Omega \mathrm{Div}\{\mu[I_2(v)]\mathcal{E}(v)\}v dx =$$
$$= 2\int_\Omega \mu[I_2(v)]\mathcal{E}(v) : \mathrm{Grad}\, v dx =$$
$$= 4\int_\Omega \mu[I_2(v)]\mathcal{E}(v) : \mathcal{E}(v) dx \geq \tag{6}$$
$$\geq 2\min_{s\geq 0}\mu(s)\|v(t,\cdot)\|^2_{\overset{\circ}{D}(\Omega)} = C_2\|v(t,\cdot)\|^2_{\overset{\circ}{D}(\Omega)}.$$

Here we made use the Corn identity: $\frac{1}{2}\|v(t,\cdot)\|^2_{\overset{\circ}{D}(\Omega)} = \|\mathcal{E}(t,\cdot)\|^2_{L_2(\Omega)}$. By applying the Cauchy inequality we get from (5), taking into account (6), the following inequality for almost all $t \geq \tau$:

$$\frac{1}{2}\cdot\frac{d}{dt}\|v(t,\cdot)\|^2_{L_2(\Omega)} + (\nu + C_2)\|v(t,\cdot)\|^2_{\overset{\circ}{D}(\Omega)} \leq \leq \|f\|_{L_2(\Omega)}\cdot\|v(t,\cdot)\|_{L_2(\Omega)}. \tag{6′}$$

whence, with regard to the properties of the Stokes operator, we obtain the inequality:

$$\frac{1}{2}\frac{d}{dt}\|v(t,\cdot)\|^2_{L_2(\Omega)} + \lambda_1(\nu + C_2)\|v(t,\cdot)\|^2_{L_2(\Omega)} \leq \leq \|f\|_{L_2(\Omega)}\cdot\|v(t,\cdot)\|_{L_2(\Omega)}.$$

Solving this inequality on the segment $[\tau,t]$ we get the estimate

$$\|v(t,\cdot)\|_{L_2(\Omega)} \leq \|v^\tau\|_{L_2(\Omega)}\exp\{-\lambda_1(\nu + C_2)(t-\tau)\} + + \frac{\|f\|_{L_2(\Omega)}}{\lambda_1(\nu + C_2)}\cdot(1-\exp\{-\lambda_1(\nu + C_2)(t-\tau)\}),\ \text{for all } t\geq\tau. \tag{7}$$

We denote $\|v^\tau\|_{L_2(\Omega)} = R_\tau$ and rewrite (7) in the form

$$\|v(t,\cdot)\|_{L_2(\Omega)} \leq R_\tau\exp\{-\lambda_1(\nu + C_2)(t-\tau)\} + +R_0(1-\exp\{-\lambda_1(\nu + C_2)(t-\tau)\}). \tag{8}$$

From (8) it follows immediately that for any $R_\tau \geq 0$ the following inequality holds

$$\|v(t,\cdot)\|_{L_2(\Omega)} \leq R_\tau + R_0,\ \text{for all } t\geq\tau. \tag{9}$$

Moreover, any trajectory which begins in the ball O_{R_τ}, gets in a finite time into the ball O_R, where $R > R_0$ and $R - R_0$ is an arbitrarily small positive number. Since $v^\tau(x)$ was fixed arbitrarily every trajectory will be found in the ball O_R, $R > R_0$, in a finite time. That means that every ball O_R, with radius which is arbitrarily close to R_0 but greater than R_0, is an absorbing set in X for the semigroup $V_{t,\tau}$. This completes the proof of Theorem 1. □

3. Energy and coercive estimates

In addition it follows from here that the semigroup $V_{t,\tau}$ is bounded and has a bounded global attractor. Integrating (6′) with respect to t and taking into account estimate (9) we get the inequality:

$$\tfrac{1}{2}\|v(t,\cdot)\|^2_{L_2(\Omega)} + (\nu + C_2)\int_\tau^t\|v(s,\cdot)\|^2_{\overset{\circ}{D}(\Omega)}ds \leq \leq \|f\|_{L_2(\Omega)}(R_\tau + R_0)(t-\tau) + \tfrac{1}{2}\|v^\tau\|^2_{L_2(\Omega)}, \tag{10}$$

whence it follows that for any solution $v(t,x)$, which begins in the ball $O_R(v(\tau,x) \in O_R)$, an energy estimate of the form

$$\max_{\tau\leq s\leq t}\|v(s,\cdot)\|^2_{L_2(\Omega)} + 2(\nu + C_2)\int_\tau^t\|v(s,\cdot)\|^2_{\overset{\circ}{D}(\Omega)}ds \leq \leq R^2 + 2\|f\|_{L_2(\Omega)}(R + R_0)(t-\tau). \tag{11}$$

is valid. We made use of the fact that for a function $v(t,x) \in W_2^{1,2}(Q)$ the norm $\|v(t,\cdot)\|_{L_2(\Omega)}$ is a continuous function of t (see, for instance, [8], Lemma 3.3).

The following lemma contains a coercive estimate of solutions of problem (2) in a form is necessary in what follows.

Lemma 1. *For solutions of problem (2) the following inequality holds:*

$$\max_{\tau\leq s\leq t}\|v(s,\cdot)\|^2_{\overset{\circ}{D}(\Omega)} + +\int_\tau^t\left[\left\|\frac{\partial v}{\partial s}\right\|^2_{L_2(\Omega)} + \|v(s,\cdot)\|^2_{W_2^2(\Omega)} + \|p(s,\cdot)\|^2_{W_2^1(\Omega)}\right]ds \leq$$
$$\leq K\left[\|v^\tau\|^2_{\overset{\circ}{D}(\Omega)} + \|f\|^2_{L_2(\Omega)}(t-\tau) + + \frac{K_2}{\varepsilon(\nu + C_2)}\left[R^2 + 2\|f\|_{L_2(\Omega)}(R + R_0)(t-\tau)\right]\right]. \tag{12}$$

The proof immediately arises from the results of [12] and from the energy estimate (11).

Theorem 2. *The family $V_{t,\tau}$ is a semigroup of compact operators i.e. $V_{t,\tau}$ is a completely continuous operator for all $t > \tau$.*

Proof. From Lemma 1 we get the inequality

$$\max_{\tau\le s\le t}\|v(s,\cdot)\|^2_{\overset{\circ}{D}(\Omega)} \le K\Bigg[\|v^\tau\|^2_{\overset{\circ}{D}(\Omega)} + \|f\|^2_{L_2(\Omega)}(t-\tau) + \\ + \frac{K_2}{\varepsilon(\nu + C_2)}\Big[R^2 + 2\|f\|_{L_2(\Omega)}(R+R_0)(t-\tau)\Big]\Bigg], \quad (13)$$

which is valid for the solution $v(t,x)$ of problem (2) for any $t \ge \tau$. Now we integrate inequality (13) with respect to τ along the trajectory of solution $v(t,x)$ taking into account the semigroup property of $V_{t,\tau}$. We note beforehand that the function

$$\psi(\tau) = \max_{\tau\le s\le t}\|v(s,\cdot)\|^2_{\overset{\circ}{D}(\Omega)}$$

is a nonincreasing nonnegative function of τ. Therefore we have the following lower bound

$$\int_\tau^t \psi(s_1)ds_1 = \int_\tau^t \max_{\tau\le s\le t}\|v(s,\cdot)\|^2_{\overset{\circ}{D}(\Omega)}\,ds_1 \ge \psi(t)(t-\tau) = \\ = \|v(t,\cdot)\|^2_{\overset{\circ}{D}(\Omega)}(t-\tau), \ \ \forall t \ge \tau. \quad (14)$$

On the right-hand side of (13) we have to integrate the function $\|v^\tau\|^2_{\overset{\circ}{D}(\Omega)}$ with respect to τ. To do this we use inequality (10). Since the right-hand side of inequality (10) does not depend on t and $\|v(t,\cdot)\|_{L_2\Omega}$ is continuous with respect to t we can rewrite (10) in the form

$$\tfrac{1}{2}\left\{\max_{\tau\le s\le t}\|v(s,\cdot)\|^2_{L_2(\Omega)} - \|v^\tau\|^2_{L_2(\Omega)}\right\} + (\nu + C_2)\int_\tau^t\|v(s,\cdot)\|^2_{\overset{\circ}{D}(\Omega)} \le \\ \le \|f\|_{L_2(\Omega)}(R+R_0)(t-\tau), \ \ \forall t \ge \tau, \quad (15)$$

where it is taken into account that $v^\tau(x) \in O_R$. From (15), by virtue of the nonnegativity of the first term on the left-hand side, we obtain the estimate

$$\int_\tau^t\|v(s,\cdot)\|^2_{\overset{\circ}{D}(\Omega)}\,ds \le \frac{\|f\|_{L_2(\Omega)}(R+R_0)}{\nu + C_2}(t-\tau), \ \ \forall t \ge \tau, \quad (16)$$

In other words we have the following estimate for the integral of function $\|v^\tau\|^2_{\overset{\circ}{D}(\Omega)}$ along the solution trajectory

$$\int_\tau^t\|v^{s_1}\|^2_{\overset{\circ}{D}(\Omega)}\,ds_1 \le \frac{\|f\|_{L_2(\Omega)}(R+R_0)}{\nu + C_2}(t-\tau), \ \ \forall t \ge \tau, \quad (17)$$

Integrating now inequality (13) with respect to τ along the trajectory of solution of problem (2) and taking into account (14) and (17), we obtain the following inequality

$$\|v(t,\cdot)\|^2_{\overset{\circ}{D}(\Omega)} \le K\Bigg[\frac{\|f\|_{L_2(\Omega)}(R+R_0)}{\nu + C_2} + \frac{\|f\|^2_{L_2(\Omega)}(t-\tau)}{2} + \\ + \frac{K_2R^2}{\varepsilon(\nu + C_2)} + \frac{K_2\|f\|_{L_2(\Omega)}(R+R_0)}{\varepsilon(\nu + C_2)}(t-\tau)\Bigg], \ \ \forall t \ge \tau. \quad (18)$$

The estimate (18) shows that the operator $V_{t,\tau}$ for any fixed $t > \tau$ maps any bounded set of initial data from the ball $O_R \subset X$ into a bounded set in $\overset{\circ}{D}(\Omega)$ which is precompact in X by virtue of the embedding theorems. Thus, operators $V_{t,\tau}$ are completely continuous for any $t > \tau$, and Theorem 2 is proved. □

From Theorems 1, 2 and the results by O. A. Ladyzhenskaya ([10], Theorem 2.1) we obtain immediately

Corollary 1. *For system (2) there exists a minimal global attractor $\mathfrak{M} \subset X$. Here $\mathfrak{M}$ is nonempty compactum which is equal to the intersection of a centered directedness of compacta, i.e.*

$$\mathfrak{M} = \bigcap_{\forall t\ge\tau}\overline{V_{t,\tau}(O_R)} \ \ (R > R_0), \quad (19)$$

where the over-line denotes the closure in topology of the phase space X, i.e. in the $L_2(\Omega)$-norm.

4. Extension of the semigroup $V_{t,\tau}$ to a group $V_{t,\tau}$

Now we proceed to the analysis of the problem of extension of the semigroup $V_{t,\tau}$ to a group $V_{t,\tau}$ for any $t \in \mathbb{R}$, $\tau \in \mathbb{R}$.

We choose initial data v^τ and $f(x)$ so that the solution $v(t,x)$ of system (2) were smooth enough: $v(t,x)$, $\dfrac{\partial v_i}{\partial x_j}, \dfrac{\partial^2 v_i}{\partial\partial x_j}$ $(i,j = 1,3)$ are continuous with respect to t. This is always possible by virtue of the Lemma on smoothness raising [1] and of the embedding theorems for spaces $W_p^{q,r}$. We denote by B the collection of such sufficiently regular initial data from the ball O_R.

We say that system (2) is well-defined in the sense of S. G. Krein [7] if Cauchy problems for the system are uniquely solvable on any segment $[\tau,T]$ for initial data $v(\tau,x) = v^{\tau}(x)$ and $v(T,x) = v^{T}(x)$.

Lemma 2. *Let $v^{\tau}(x) \in B$. Then system (2) is well-defined in the sense of S. G. Krein.*

Proof. Projecting (2) on $H(\Omega)$ we get

$$\frac{\partial v}{\partial t} + \Pi\left(\frac{v}{1+\varepsilon|v|^2}\operatorname{grad}\right)v -$$

$$-\Pi\operatorname{Div}[(\nu + 2\mu[I_2(v)])I\operatorname{Grad} v] = \Pi f, \quad (20)$$

$$t \in [\tau,T];\ v(\tau,x) = v^{\tau}(x).$$

Here Π is an orthoprojector, $\Pi : L_2(\Omega) \to H(\Omega)$. In (2) $u = u(t,x)$ is already known solution of problem (20) which has the regularity indicated above by virtue of the definition of B.

The quadratic form $((\nu + 2\nu[I_2(u)])I\xi,\xi) \geq \nu(\xi,\xi)$ is positive definite by virtue of condition (4) (ii), item 3: for an equation with a selfadjoint second order operator with coefficients having continuous partial derivatives with respect to t the problem

$$\frac{\partial u}{\partial t} + \sum_{i,j=1}^{n}(a_{ij}(t,x)\frac{\partial u}{\partial x_i}) = 0,\ u|_{\partial\Omega} = 0,\ u(0,x) = u^0(x)$$

is well-posed in the class of solutions which are uniformly with respect to t bouded in $L_2(\Omega)$, if for all $(t,x) \in [\tau,T] \times \bar{\Omega}$ the quadratic form $\Sigma a_{ij}\xi_i\xi_j$ is of constant signs, and of O. A. Ladyzhenskaya ([9], § 2), where results from [9] are extended to the case of a not selfajoint operator connected with Navier-Stokes equations and general quasilinear equations of parabolic type, we obtain that the remainder of two solutions of problem (20) $v^1(t,x) - v^2(t,x)$ has the following estimate

$$\left\|v^1(t,\cdot) - v^2(t,\cdot)\right\|_{L_2(\Omega)} \leq$$

$$\leq \left\|v^1(\tau,\cdot) - v^2(\tau,\cdot)\right\|_{L_2(\Omega)}^{1-\alpha(t)} \cdot \left\|v^1(T,\cdot) - v^2(T,\cdot)\right\|_{L_2(\Omega)}^{\alpha(t)}$$

for all $t \in [\tau,T]$ and some function $\alpha(t) : 0 < \alpha(t) < 1$. Thus, Lemma 2 is proved. □

Corollary 2. *The attractor $\mathfrak{M}_1 = \bigcap_{\forall t\geq\tau}\overline{V_{t,\tau}(B)}$ consists only of whole trajectories v(t,x) of system (2). The semigroup $V_{t,\tau}$ $(t \geq \tau)$ on $\mathfrak{M}_1$ extends to a group of nonlinear operators $V_{t,\tau}$, for all $t, \tau \in \mathbb{R}$.*

The proof immediately arises from Corollary 1, Lemma 2 and the results of O. A. Ladyzhenskaya ([10], Theorem 2.5, remark 2). Thus, the group $V_{t,\tau}$ assigns some dynamic system on $\mathfrak{M}_1$.

We remarc that the attractor $\mathfrak{M}_1$ obtained above consist only of those trajectories of system (2) on which the considered evolutionary problem $(t \geq \tau)$ can be uniquely extended to a dynamical system $(t \in \mathbb{R})$ with the phase set $\mathfrak{M}_1$. In this sense we obtain an attractor which is analogous to the minimal global attractor for two-dimensional Navier-Stokes equations.

Lemma 3. *The set $\mathfrak{M}_1 = \bigcap_{\forall t\geq\tau}\overline{V_{t,\tau}(B)}$ is bounded in the $\overset{\circ}{D}(\Omega)$-norm.*

Proof. By virtue of the definition of B, $\mathfrak{M}_1 \subset O_R$ $(R > R_0)$. We fix an arbitrary whole trajectory $v(x,x)$ $(t \in \mathbb{R},\ x \in \bar{\Omega}$ from $\mathfrak{M}_1$. For $v(t,x)$ we can take as $v(\tau,x)$ any point of the trajectory, in particular, for an arbitrarily large negative τ $(\tau \to -\infty)$. Therefore we derive from inequality (18)

$$\|v(t,\cdot)\|_{\overset{\circ}{D}(\Omega)}^2 \leq \min_{\forall\tau\leq t}\left\{K\left[\frac{\|f\|_{L_2(\Omega)}(R+R_0)}{\nu+C_2} + \frac{\|f\|_{L_2(\Omega)}^2}{2}(t-\tau) + \right.\right.$$

$$\left.\left. + \frac{K_2R^2}{\varepsilon(\nu+C_2)} + \frac{K_2\|f\|_{L_2(\Omega)}(R+R_0)}{\varepsilon(\nu+C_2)}(t-\tau)\right]\right\} =$$

$$= K\left[\frac{\|f\|_{L_2(\Omega)}(R+R_0)}{\nu+C_2} + \frac{K_2R^2}{\varepsilon(\nu+C_2)}\right] \equiv \beta^2,\ \forall t \in \mathbb{R}. \quad (21)$$

As we see from (21) the quantity $\beta = \beta(\frac{1}{\varepsilon})$ does not depend on the choice of the trajectory $v(t,x) \in \mathfrak{M}_1$. But by virtue of Corollary 2 $\mathfrak{M}_1$ consists only of whole trajectories. Thus, the derived estimate is valid for all points of $\mathfrak{M}_1$. This completes the proof of Lemma 3. □

5. Dynamics finite dimensionality of $V_{t,\tau}$ group on $\mathfrak{M}_1$

We remind that dynamics finite dimensionality of V_t, $t \in \mathbb{R}$ on attractors [10] means that there exists a positive integer N such that any whole trajectory on $\mathfrak{M}_1$ is uniquely determined by its orthoprojection on some N-dimensional subspace of the phase space X.

Theorem 3. *The group $V_{t,\tau}$ possesses the property of dynamics finite dimensionality on $\mathfrak{M}_1$.*

PROOF. Let $\{u,q\}$ and $\{v,p\}$ be two solutions of problem (2). Denote $z(t,x) = u(t,x) - v(t,x)$. Then we obtain for $z(t,x)$ the following relations

$$\begin{aligned}&\frac{\partial z}{\partial t} + D_\varepsilon(u) - D_\varepsilon(v) + \nu \operatorname{rot}\operatorname{rot} z + B(u) - B(v) + \\ &+ \operatorname{grad}(q-p) = 0, \ \operatorname{div} z = 0, \ (t \ge \tau, x \in \Omega); \\ &z = 0 \ (t \ge \tau, x \in \partial\Omega); \\ &z(\tau, x) = u(\tau,x) - v(\tau,x) \ x \in \overline{\Omega}; \\ &(q-p,1)_{L_2(\Omega)} = 0; \ \varepsilon > 0.\end{aligned} \tag{22}$$

Multiplying the first equation of (22) by $z(t,x)$ scalarly in $L_2(\Omega)$ we arrive to equality

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|z(t,\cdot)\|^2_{L_2(\Omega)} + \nu\|z(t,\cdot)\|^2_{\overset{\circ}{D}(\Omega)} + (B(u)-B(v),z)_{L_2(\Omega)} = \\ &\qquad = -(D_\varepsilon(u) - D_\varepsilon(v), z)_{L_2(\Omega)}.\end{aligned} \tag{23}$$

Lagrange's mean value theorem ($\delta \in [0,1]$, δ_0 is a fixed value δ) and condition (4) (i) allow us to prove the monotonicity of operator $B(u)$ ([1], pp. 113—114)

From (23) in virtue of the proved inequality (24) we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|z(t,\cdot)\|^2_{L_2(\Omega)} + (\nu + C_3)\|z(t,\cdot)\|^2_{\overset{\circ}{D}(\Omega)} \le \\ &\qquad \le \left|(D_\varepsilon(u) - D_\varepsilon(v), z)_{L_2(\Omega)}\right|.\end{aligned} \tag{25}$$

We estimate now from above the right-hand side of inequality (25). As $z(t,x)$ is solenoidal we get the following chain of inequalities:

$$\begin{aligned}&\left|(D_\varepsilon(u) - D_\varepsilon(v), z)_{L_2(\Omega)}\right| \le \\ &\le \left|\left(\frac{u\times \operatorname{rot} u - v \times \operatorname{rot} v}{(1+\varepsilon|u|^2)(1+\varepsilon|v|^2)}, u - v\right)_{L_2(\Omega)}\right| + \\ &+\varepsilon\left|\left(\frac{|u|^2 \operatorname{rot} u - |v|^2 \operatorname{rot} v}{(1+\varepsilon|u|^2)(1+\varepsilon|v|^2)}, u\times v\right)_{L_2(\Omega)}\right| = \\ &= \left|\left(\frac{(u+v)\times \operatorname{rot}(u-v)}{(1+\varepsilon|u|^2)(1+\varepsilon|v|^2)}, u - v\right)_{L_2(\Omega)}\right| +\end{aligned}$$

$$\begin{aligned}&(B(u)-B(v),z)_{L_2(\Omega)} = -2\int_\Omega \operatorname{Div}\{\mu[I_2(u)]\mathcal{E}(u) - \mu[I_2(v)]\mathcal{E}(v)\} z\,dx = \\ &= 2\int_\Omega \{\mu[I_2(u)]\mathcal{E}(u) - \mu[I_2(v)]\mathcal{E}(v)\} : \operatorname{Grad} z dx = \\ &= 4\int_\Omega \{\mu[I_2(u)]\mathcal{E}(u) - \mu[I_2(v)]\mathcal{E}(v)\} : \mathcal{E}(z) dx = \\ &= 4\int_\Omega \frac{d}{d\delta}\{\mu[I_2(v+\delta_0 z)]\mathcal{E}[v+\delta_0 z] : \mathcal{E}(z)dx = \\ &= 4\int_\Omega \{\mu[I_2(v+\delta_0 z)]\mathcal{E}(z) + \frac{d}{d\delta}\mu[I_2(v+\delta_0 z)]\mathcal{E}[v+\delta_0 z]\} : \mathcal{E}(z)dx = \\ &= 4\int_\Omega \{\mu[I_2(v+\delta_0 z)]\mathcal{E}(z) : \mathcal{E}(z) + \frac{d}{d\delta}\mu\left([\mathcal{E}[v+\delta_0 z] : \mathcal{E}[v+\delta_0 z])^{1/2}\right]\mathcal{E}[v+\delta_0 z] : \mathcal{E}(z)\}dx = \\ &= 4\int_\Omega \left\{\mu[I_2(v+\delta_0 z)]\mathcal{E}(z) : \mathcal{E}(z) + \frac{\mathcal{E}[v+\delta_0 z] : \mathcal{E}(z)}{(\mathcal{E}[v+\delta_0 z] : \mathcal{E}[v+\delta_0 z])^{1/2}} \cdot \frac{d\mu[I_2(v+\delta_0 z)]}{dI_2}\mathcal{E}[v+\delta_0 z] : \mathcal{E}(z)\right\}dx = \\ &= 4\int_\Omega \left\{\mu[I_2(v+\delta_0 z)]\mathcal{E}(z) : \mathcal{E}(z) + \frac{1}{I_2(v+\delta_0 z)} \cdot \frac{d\mu[I_2(v+\delta_0 z)]}{dI_2}[\mathcal{E}[v+\delta_0 z] : \mathcal{E}(z)]^2\right\}dx \ge \\ &\ge \left\{\begin{array}{ll} 4\min\limits_s \mu(s)\|\mathcal{E}(z)\|^2_{L_2(\Omega)} \ge 2C_2\|z(t,\cdot)\|^2_{\overset{\circ}{D}(\Omega)}, & \text{if } \dfrac{d\mu(s)}{ds} > 0; \\ 4\min\limits_s\left(\mu(s) + s\dfrac{d\mu}{ds}\right)\cdot\|\mathcal{E}(z)\|^2_{L_2(\Omega)} \ge 2C_1\|z(t,\cdot)\|^2_{\overset{\circ}{D}(\Omega)}, & \text{if } \dfrac{d\mu(s)}{ds} \le 0; \end{array}\right\} \ge C_3\|z\|^2_{\overset{\circ}{D}(\Omega)}, \\ &C_3 = 2\min\{C_1, C_2\}.\end{aligned} \tag{24}$$

$$+\varepsilon\left|\left(\frac{(v+u)(v-u)\operatorname{rot}u}{(1+\varepsilon|u|^2)(1+\varepsilon|v|^2)},u\times v\right)_{L_2(\Omega)}+\right.$$
$$\left.+\left(\frac{|u|^2\operatorname{rot}(u-v)}{(1+\varepsilon|u|^2)(1+\varepsilon|v|^2)},u\times v\right)_{L_2(\Omega)}\right|\le$$
$$\le\frac{1}{\sqrt{\varepsilon}}\|z(t,\cdot)\|_{\overset{\circ}{D}(\Omega)}\cdot\|z(t,\cdot)\|_{L_2(\Omega)}+$$
$$+\varepsilon\left|\left(\frac{(v+u)(v-u)\operatorname{rot}u}{(1+\varepsilon|u|^2)(1+\varepsilon|v|^2)},(u-v)\times v\right)_{L_2(\Omega)}\right|+$$
$$+\varepsilon\left|\left(\frac{|u|^2\operatorname{rot}(u-v)}{(1+\varepsilon|u|^2)(1+\varepsilon|v|^2)},(u-v)\times v\right)_{L_2(\Omega)}\right|\le$$
$$\le\frac{2}{\sqrt{\varepsilon}}\|z(t,\cdot)\|_{\overset{\circ}{D}(\Omega)}\cdot\|z(t,\cdot)\|_{L_2(\Omega)}+\frac{3}{2}\|u(t,\cdot)\|_{\overset{\circ}{D}(\Omega)}\|z(t,\cdot)\|^2_{L_4(\Omega)}. \tag{26}$$

From (25) taking into account (26) we obtain:

$$\frac{1}{2}\frac{d}{dt}\|z(t,\cdot)\|^2_{L_2(\Omega)}+(\nu+C_3)\|z(t,\cdot)\|^2_{\overset{\circ}{D}(\Omega)}\le$$
$$\le\frac{2}{\sqrt{\varepsilon}}\|z(t,\cdot)\|_{\overset{\circ}{D}(\Omega)}\cdot\|z(t,\cdot)\|_{L_2(\Omega)}+\frac{3}{2}\|u(t,\cdot)\|_{\overset{\circ}{D}(\Omega)}\|z(t,\cdot)\|^2_{L_4(\Omega)}. \tag{27}$$

Let now $u(t,x)$ and $v(t,x)$ be whole trajectories from $\mathfrak{M}_1$. From Lemma 3 arises that $\|u(t,\cdot)\|_{\overset{\circ}{D}(\Omega)}\le\beta$ for all $t\in\mathbb{R}$. Thus, inequality (27) takes the form

$$\frac{1}{2}\frac{d}{dt}\|z(t,\cdot)\|^2_{L_2(\Omega)}+(\nu+C_3)\|z(t,\cdot)\|^2_{\overset{\circ}{D}(\Omega)}\le$$
$$\le\frac{2}{\sqrt{\varepsilon}}\|z(t,\cdot)\|_{\overset{\circ}{D}(\Omega)}\cdot\|z(t,\cdot)\|_{L_2(\Omega)}+\frac{3}{2}\beta\|z(t,\cdot)\|^2_{L_4(\Omega)}. \tag{28}$$

Using the well-known inequalities of Young and Ladyzhenskaya we get from (28):

$$\frac{1}{2}\frac{d}{dt}\|z(t,\cdot)\|^2_{L_2(\Omega)}+(\nu+C_3)\|z(t,\cdot)\|^2_{\overset{\circ}{D}(\Omega)}\le$$
$$\le\frac{(\nu+C_3)}{4}\|z(t,\cdot)\|^2_{\overset{\circ}{D}(\Omega)}+\frac{8\|z(t,\cdot)\|^2_{L_2(\Omega)}}{\varepsilon(\nu+C_3)}+$$
$$+\sqrt[4]{12}\beta\|z(t,\cdot)\|^{3/2}_{\overset{\circ}{D}(\Omega)}\cdot\|z(t,\cdot)\|^{1/2}_{L_2(\Omega)}\le$$
$$\le\frac{(\nu+C_3)}{2}\|z(t,\cdot)\|^2_{\overset{\circ}{D}(\Omega)}+$$
$$+\left[\frac{8}{\varepsilon(\nu+C_3)}+\frac{81\beta^4}{(\nu+C_3)^3}\right]\|z(t,\cdot)\|^2_{L_2(\Omega)}.$$

Hence we obtain the required differential inequality

$$\frac{1}{2}\frac{d}{dt}\|z(t,\cdot)\|^2_{L_2(\Omega)}+\frac{(\nu+C_3)}{2}\|z(t,\cdot)\|^2_{\overset{\circ}{D}(\Omega)}\le$$
$$\le\frac{1}{(\nu+C_3)}\left[\frac{8}{\varepsilon}+\frac{81\beta^4}{(\nu+C_3)^2}\right]\|z(t,\cdot)\|^2_{L_2(\Omega)}. \tag{29}$$

Let now the equality $P_Nu = P_Nv$ be valid for some positive integer N. Taking into account the permutability of operators P_N and $A^{1/2}$, and also the equivalence of the norm $\|\cdot\|_{\overset{\circ}{D}(\Omega)}$ and $\|A^{1/2}(\cdot)\|_{L_2(\Omega)}$, we get from (29):

$$\frac{1}{2}\frac{d}{dt}\|Q_Nz(t,\cdot)\|^2_{L_2(\Omega)}+\frac{(\nu+C_3)}{2}\|A^{1/2}z(t,\cdot)\|^2_{L_2(\Omega)}\le$$
$$\le\frac{C}{\nu+C_3}\left[\frac{8}{\varepsilon}+\frac{81\beta^4}{(\nu+C_3)^2}\right]\|Q_Nz(t,\cdot)\|^2_{L_2(\Omega)}.$$

Hence it follows that

$$\frac{1}{2}\frac{d}{dt}\|Q_Nz(t,\cdot)\|^2_{L_2(\Omega)}+\lambda_{N+1}\frac{(\nu+C_3)}{2}\|Q_Nz(t,\cdot)\|^2_{L_2(\Omega)}\le$$
$$\le\frac{C}{\nu+C_3}\left[\frac{8}{\varepsilon}+\frac{81\beta^4}{(\nu+C_3)^2}\right]\|Q_Nz(t,\cdot)\|^2_{L_2(\Omega)}. \tag{30}$$

Now we choose a number N_1 so large that

$$\lambda_{N_1+1}\frac{(\nu+C_3)}{2}-\frac{C}{\nu+C_3}\left[\frac{8}{\varepsilon}+\frac{81\beta^4}{(\nu+C_3)^2}\right]\equiv M>0,$$

then from (30) we deduce the inequality

$$\frac{1}{2}\frac{d}{dt}\|Q_{N_1}z(t,\cdot)\|^2_{L_2(\Omega)}+M\|Q_{N_1}z(t,\cdot)\|^2_{L_2(\Omega)}\le 0,$$

by solving it on the segment $[\tau,t]$ we obtain

$$\|Q_{N_1}z(t,\cdot)\|_{L_2(\Omega)}\le e^{-M(t-\tau)}\|Q_{N_1}z(\tau,\cdot)\|_{L_2(\Omega)},\quad \forall t\ge\tau,\forall\tau\in\mathbb{R}. \tag{31}$$

In virtue of Theorem 2 both trajectories $u(t,x)$ and $v(t,x)$ belong wholly to the ball O_R. Therefore we get from (31) the inequality

$$\|Q_{N_1}z(t,\cdot)\|_{L_2(\Omega)}\le 2e^{-M(t-\tau)}\cdot R,$$

which holds for all $t\ge\tau$ for any $\tau\in\mathbb{R}$. Letting τ tend to $-\infty$ we obtain $\|Q_{N_1}z(t,\cdot)\|_{L_2(\Omega)}\equiv 0$ for all $t\in\mathbb{R}$, i.e. $Q_{N_1}u = Q_{N_1}v$, and thus Theorem 3 is proved. □

6. Hausdorff dimension of $\mathfrak{M}_1$

Theorem 4. *The set $\mathfrak{M}_1$ has a finite Hausdorff dimension.*

Proof. We denote again by $u(t,x)$ and $v(t,x)$ two arbitrary whole trajectories from $\mathfrak{M}_1$, $z(t,x) = u(t,x) - v(t,x)$. To simplify the notation we will consider the problem on the segment $[0,T]$. From inequality (29) we deduce

$$\|z(t,\cdot)\|_{L_2(\Omega)} \le \exp\left\{\frac{T}{\nu+C_3}\left[\frac{8}{\varepsilon}+\frac{81\beta^4}{(\nu+C_3)^2}\right]\right\}\|z(0,\cdot)\|_{L_2(\Omega)} \equiv$$
$$\equiv F_1 \cdot \|z(0,\cdot)\|_{L_2(\Omega)}. \qquad (32)$$

From (29) taking into account (32) we obtain the following differential inequality

$$\frac{1}{2}\frac{d}{dt}\|z(t,\cdot)\|^2_{L_2(\Omega)} + \frac{(\nu+C_3)}{2}\|z(t,\cdot)\|^2_{\overset{\circ}{D}(\Omega)} \le$$
$$\le \frac{1}{(\nu+C_3)}\left[\frac{8}{\varepsilon}+\frac{81\beta^4}{(\nu+C_3)^2}\right]F_1^2\,\|z(0,\cdot)\|^2_{L_2(\Omega)} \equiv \qquad (33)$$
$$\equiv F_2 \cdot \|z(0,\cdot)\|^2_{L_2(\Omega)}.$$

We integrate (33) from 0 to t and derive

$$\frac{1}{2}\|z(t,\cdot)\|^2_{L_2(\Omega)} + \frac{(\nu+C_3)}{2}\int_0^t\|A^{1/2}z(s,\cdot)\|^2_{L_2(\Omega)}ds \le$$
$$\le C\left(\frac{1}{2}+F_2t\right)\|z(0,\cdot)\|^2_{L_2(\Omega)}. \qquad (34)$$

In virtue of the permutability of the ortho-projectors P_N, Q_N and the operator $A^{1/2}$ we have

$$\|A^{1/2}z(t,\cdot)\|^2_{L_2(\Omega)} = \|A^{1/2}(P_N+Q_N)z(t,\cdot)\|^2_{L_2(\Omega)} =$$
$$= \|P_NA^{1/2}z(t,\cdot)+Q_NA^{1/2}z(t,\cdot)\|^2_{L_2(\Omega)} =$$
$$= \|P_NA^{1/2}z(t,\cdot)\|^2_{L_2(\Omega)} + \|Q_NA^{1/2}z(t,\cdot)\|^2_{L_2(\Omega)} =$$
$$= \|A^{1/2}P_Nz(t,\cdot)\|^2_{L_2(\Omega)} + \|A^{1/2}Q_Nz(t,\cdot)\|^2_{L_2(\Omega)} \ge$$
$$\ge \|A^{1/2}Q_Nz(t,\cdot)\|^2_{L_2(\Omega)}.$$

Moreover, it is evident that $\|z(t,\cdot)\|_{L_2(\Omega)} \ge$ $\ge \|Q_Nz(t,\cdot)\|_{L_2(\Omega)}$, therefore we obtain from (34) the inequality

$$\frac{1}{2}\|Q_Nz(t,\cdot)\|^2_{L_2(\Omega)} + \frac{(\nu+C_3)}{2}\int_0^t\|A^{1/2}Q_Nz(s,\cdot)\|^2_{L_2(\Omega)}ds \le$$
$$\le C\left(\frac{1}{2}+F_2t\right)\|z(0,\cdot)\|^2_{L_2(\Omega)},$$

from which we derive an integral inequality of the form

$$\frac{1}{2}\|Q_Nz(t,\cdot)\|^2_{L_2(\Omega)} + \lambda_{N+1}\frac{(\nu+C_3)}{2}\int_0^t\|Q_Nz(s,\cdot)\|^2_{L_2(\Omega)}ds \le$$
$$\le C\left(\frac{1}{2}+F_2t\right)\|z(0,\cdot)\|^2_{L_2(\Omega)}, \qquad (35)$$

which is valid for any $t \in [0,T]$. From (35) we get immediately the estimate

$$\|Q_Nz(t,\cdot)\|^2_{L_2(\Omega)} \le \Big\{\exp[-\lambda_{N+1}(\nu+C_3)t] +$$
$$+\frac{2F_2(1-\exp[-\lambda_{N+1}(\nu+C_3)t])}{\lambda_{N+1}(\nu+C_3)}\Big\}\|z(0,\cdot)\|^2_{L_2(\Omega)} \equiv \qquad (36)$$
$$\equiv \delta^2(t,N)\|z(0,\cdot)\|^2_{L_2(\Omega)},$$

which shows that there exists a so large $N = N_2$ that for some $t_1 \in [0,T]$ the condition $\delta(t_1,N_2) < 1$ is fulfilled besides simultaneously for all arbitrary couples of whole trajectories $u(t,x)$ and $v(t,x)$ from $\mathfrak{M}_1$. Thus, we find ourselves in conditions of applicability of the abstract theorem of O. A. Ladyzhenskaya ([10], Theorem 2.8), from which we immediately obtain the required assertion. Theorem 4 is proved. □

7. Conclusions

At the beginning of this article we have already noted that obtained inequalities are only upper estimates. Now we can say some more. Really, from definition of an attractor $\mathfrak{M}_1$ it is clear that this set contains all solutions of the stationary problem. In [12] for this problem upper estimates which depend only on $\|f\|_{L_2(\Omega)}$ and do not depend on ε were obtained. Consequently, for an attractor as a whole there do not exist lower bounds which depend on ε. It is clear that all trajectories of periodic solutions which existence was proved [13] form a subset of $\mathfrak{M}_1$. However, in [13] the estimate $\max\limits_{0\le t\le 1}\|v(t,\cdot)\|^2_{\overset{\circ}{D}(\Omega)} \le M(\varepsilon)\|f\|^2_{L_2(\Omega)}$ was obtained, where $M(\varepsilon)$ is a singular function of ε. Thus, we can presuppose that stationary solutions form a very “little” subset of $\mathfrak{M}_1$, while to an attractor as a whole basic qualities of the collection of *periodic* solutions of the problem are intrinsic.

REFERENCES

1. Agranovich Yu. Ya. Mathematical models of linear and nonlinear visco-elastic fluids. Ph.D. Dissert.

written under the supervision of Prof. P. E. Sobolevskii at the Voronezh State University. — Voronezh. 1991.

2. Agranovich Yu. Ya. On the influence of the inertia parameter on the number of the defining modes and the fractal dimension of an attractor for one regularization of Navier-Stokes equations // IV Workshop on PDE, IMPA, July 17—21: Abtsracts. Rio de Janeiro, 1995.

3. Astarita G., Marrucci G.. Principless of Non-Newtonian Fluid Mechanics// McGrow-Hill, 1947.

4. Cotsiolis A. A., Oskolkov A. P. On the marginal behavior and on the attractor for the equations of motion of Oldroyd's fluid // Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI). — 1986. — V. **152**, — P. 67—71 (Russ.).

5. Karazeeva N. A., Cotsiolis A. A., Oskolkov A. P. Dynamic systems and attractors generated by initial boundary value problems for equations of linear visco-elastic fluids // Trudy Mat. Inst. Steklov. — 1990. — V. **188**. — P. 60—87 (Russ.).

6. Korn G. A., Korn Th. M. Mathematical handbook for scientists and engineers. — New York, 1961.

7. Krein S. G. On classes of well-posedness for some initial boundary value problems // Dokl. Akad. Nauk USSR. — 1957. — V. **114**. — P. 1162—1165 (Russ.).

8. Ladyzhenskaya O. A., Solonnikov V. A., Ural'tseva N. N. Linear and quasilinear equations of parabolic type. Moscow// Amer. Math. Soc. — 1969.

9. Ladyzhenskaya O. A. On the dynamic system generated by Navier-Stokes equations // Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI). — 1972. — V. **27**. — P. 91—114 (Russ.).

10. Ladyzhenskaya O. A. On determination of a minimal global attractors for Navier-Stokes equations and for other partial differential equations // Uspekhi Mat. Nauk. — 1987. — V. **42**, № **6**, — P. 25—60 (Russ.).

11. Litvinov W. G. The motion of a nonlinear viscous fluid. — Moscow: Nauka, — 1987.

12. Sobolevskii P. E. An existence of solutions of a mathematical model of nonlinear viscous fluid // Dokl. Akad. Nauk USSR. — 1985. — V. **285**, — P. 44—48; English transl. in Soviet Math. Dokl. **32**.

13. Sobolevskii P. E. Periodic motions of a nonlinear viscous fluid. Qualitative of differential equations theory// Siberian Otdel. Akad. Nauk USSR. — Irkutsk, — P. 128—134.

14. Temam R. Une Methode d'Approximation de la Solution des Equations de Navier-Stokes // Bull. Soc. Math. France. — 1968. — V. 98. — P. 115—152.

15. Truesdell G., Noll W. The Non-Linear Field Theories of Mechanics. — Berlin: Springer-Verlag, — 1965.