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VISCOUS HYDRODYNAMICS THROUGH STOCHASTIC PERTURBATIONS OF FLOWS OF PERFECT FLUIDS ON GROUPS OF DIFFEOMORPHISMS*

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We construct some stochastic perturbations of the curves on groups of H^s ($s > \frac{1}{2}n+1$) Sobolev diffeomorphisms of a flat n-dimensional torus, describing the motion of diffuse matter or perfect incompressible fluid, such that their expectations describe the dynamics of corresponding viscous fluids. Existence of classical solutions for initial value problem of Burgers and Navier-Stokes equations for $s > \frac{1}{2}n+2$ is proved (for $s > \frac{1}{2}n+1$ — of some sort of generalized solutions).

1. Introduction and preliminaries

In this paper we construct some special stochastic perturbations of the curves on the groups of diffeomorphisms, describing the motion of perfect fluids, such that the expectations of obtained processes describe the motion of viscous fluids. This yields existence of solutions of initial value problem for Burgers and Navier-Stokes equations with some sort of external forces (in particular, with zero forces) and clarifies relations with solutions of Hopf and Euler equations, respectively, with the same initial data. At the end of the paper we summarize the results so that the reader can find the exact formulations there.

This approach to hydrodynamics belongs to the direction suggested for perfect fluids by Arnold [1] and developed by Ebin and Marsden [5]. In [6] – [8] (see also detailed explanation in [9] and [10]) a certain stochastic analogue of the second Newton's law was discovered that expanded this geometrical approach to viscous fluids. However there was a problem to find stochastic processes satisfying the stochastic Newton's law that was investigated by various methods (see, e.g., [3]). In [4] we elaborated the idea to construct such processes via stochastic perturbations of deterministic curves. Here this idea is seriously modified so that it allows us to cover the cases mentioned above.

We consider the fluid motion on the flat n-dimensional torus $\mathcal{T}^n$ (the quotient of R^n with respect to integral lattice where the Riemannian metric is inherited from R^n). The natural configuration space for fluids, admitting compressibility, is the Hilbert manifold $\mathcal{D}^s(\mathcal{T}^n)$ of Sobolev H^s-diffeomorphisms, $s > \frac{n}{2}+1$. There is also a group structure on $\mathcal{D}^s(\mathcal{T}^n)$ with respect to the composition. The set $\mathcal{D}^s_\mu(\mathcal{T}^n)$ of H^s-diffeomorphisms preserving the volume is a submanifold and subgroup in $\mathcal{D}^s(\mathcal{T}^n)$. This is the natural configuration space for incompressible fluids. The details of geometry on those groups can be found in [5], [9] and [10]. In [4] the presentation of this material is adapted specially to the groups of diffeomorphisms of flat torus. Here we describe the preliminaries rather briefly paying main attention to some new points absent in [4].

The tangent space $T_e\mathcal{D}^s(\mathcal{T}^n)$ at $e = id$ is the set of all H^s-vector fields on $\mathcal{T}^n$ and $T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$ is the set of all divergence-free H^s-vector fields. In $T_e\mathcal{D}^s(\mathcal{T}^n)$ (and so in $T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$) one introduces the L_2 scalar product denoted by $(\cdot,\cdot)$.

The right-hand translation $R_f : \mathcal{D}^s_\mu(\mathcal{T}^n) \to \mathcal{D}^s_\mu(\mathcal{T}^n)$, $R_f \circ \theta = \theta \circ f$, $\theta, f \in \mathcal{D}^s_\mu(\mathcal{T}^n)$, is C^∞-smooth and thus one may consider right-invariant vector fields on $\mathcal{D}^s_\mu(\mathcal{T}^n)$. Notice that the tangent to right translation takes the form: $TR_fX = X \circ f$ for $X \in T\mathcal{D}^s_\mu(\mathcal{T}^n)$. For $\mathcal{D}^s(\mathcal{T}^n)$ we have analogous properties.

The left-hand translation $L_f : \mathcal{D}^s_\mu(\mathcal{T}^n) \to \mathcal{D}^s_\mu(\mathcal{T}^n)$, $L_f \circ \theta = f \circ \theta$, $\theta, f \in \mathcal{D}^s_\mu(\mathcal{T}^n)$, is not smooth but it is continuous and we shall use this property below.

A right-invariant vector field X on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ generated by a vector $X \in T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$ is C^k-smooth iff the vector field X on $\mathcal{T}^n$ is H^{s+k}-smooth (for

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$\mathcal{D}^s_\mu(\mathcal{T}^n)$ we have the same property). This fact is a consequence of the so-called ω-lemma (see [5]) and it is valid also for more complicated fields. For example, if a tensor (or any other) field on $\mathcal{T}^n$ is C^∞-smooth, the corresponding right-invariant field on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ (and on $\mathcal{D}^s(\mathcal{T}^n)$) is C^∞-smooth as well.

Also by right-hand translations of $(\cdot,\cdot)$ we determine the scalar products in all tangent spaces and so obtain the so-called weak Riemannian metric on $\mathcal{D}^s(\mathcal{T}^n)$ and on $\mathcal{D}^s_\mu(\mathcal{T}^n)$. This metric admits the Levi-Civita connection and covariant derivative in the following way. Denote by K the connector of flat connection on $\mathcal{T}^n$. For vector fields X, Y on $\mathcal{D}^s(\mathcal{T}^n)$ and for a vector field $X(t)$ along a certain smooth curve $g(t)$ in $\mathcal{D}^s(\mathcal{T}^n)$ we define the covariant derivatives $\bar\nabla_X Y$ and $\frac{\bar D}{dt}X(t)$, respectively, by formulae

$$\bar\nabla_X Y = K\circ TY(X),$$
$$\frac{\bar D}{dt}X(t) = K\circ\frac{d}{dt}X(t). \qquad (1)$$

Denote by $\bar{\mathcal{H}}$ the connection on $\mathcal{D}^s(M)$ corresponding to $\bar\nabla$.

The geodesic spray $\bar{\mathcal{Z}}$ of $\bar{\mathcal{H}}$ is the vector field $T\mathcal{D}^s(\mathcal{T}^n)$ of the form

$$\bar{\mathcal{Z}}(X) = \mathcal{Z}\circ X \qquad (2)$$

for $X\in T\mathcal{D}^s(\mathcal{T}^n)$, where $\mathcal{Z}$ is the geodesic spray of the flat connection on $\mathcal{T}^n$. From (2) one can easily see that by the construction $\bar{\mathcal{Z}}$ is $\mathcal{D}^s(\mathcal{T}^n)$-right-invariant and by ω-lemma it is C^∞-smooth on $T\mathcal{D}^s(\mathcal{T}^n)$ since $\mathcal{Z}$ is C^∞-smooth on $T\mathcal{T}^n$.

Denote by $P_e : T_e\mathcal{D}^s(\mathcal{T}^n)\to E^s\oplus ker\Delta = T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$ the $(\cdot,\cdot)$-orthogonal projection. Consider the map $P : T\mathcal{D}^s(\mathcal{T}^n)_{|\mathcal{D}^s_\mu(\mathcal{T}^n)}\to T\mathcal{D}^s_\mu(\mathcal{T}^n)$ determined for each $\eta\in\mathcal{D}^s_\mu(\mathcal{T}^n)$ by the formula

$$P_\eta = TR_\eta\circ P_e\circ TR_\eta^{-1}.$$

It is obvious that P is $\mathcal{D}^s_\mu(\mathcal{T}^n)$-right-invariant. There is an important and rather complicated result (see [5]) that P is C^∞-smooth. Since by Hodge decomposition the orthogonal complement to $T_e\mathcal{D}^s_\mu(M)$ consists of gradients, for every $Y\in T_e\mathcal{D}^s(M)$ we have

$$P_e(Y) = Y - \mathrm{grad}p \qquad (3)$$

where p is a certain H^{s+1}-function on $\mathcal{T}^n$ unique to within the constants (see also [15]).

Since $\mathcal{D}^s_\mu(\mathcal{T}^n)$ is a submanifold in $\mathcal{D}^s(\mathcal{T}^n)$, there is a corresponding standard connection $\tilde{\mathcal{H}}$ on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ whose connector $\tilde K$ and the covariant derivatives $\tilde\nabla$ and $\frac{\tilde D}{dt}$ are described by the formulae

$$\tilde K = P\circ K,$$
$$\tilde\nabla_X Y = P\circ\bar\nabla_X Y = P\circ K\circ TY(X) = \tilde K\circ TY(X)$$
$$\frac{\tilde D}{dt}X(t) = P\circ\frac{\bar D}{dt}X(t) = P\circ K\circ\frac{d}{dt}X(t) =$$
$$= \tilde K\circ\frac{d}{dt}X(t) \qquad (4)$$

where X, Y are vector fields on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ and $X(t)$ is a vector field along a certain smooth curve $g(t)$ in $\mathcal{D}^s_\mu(\mathcal{T}^n)$.

The geodesic spray $\mathcal{S}$ of $\tilde{\mathcal{H}}$ is a vector field on $T\mathcal{D}^s_\mu(\mathcal{T}^n)$ of the form

$$\mathcal{S}(X) = TP(\bar{\mathcal{Z}}\circ X),\quad X\in T\mathcal{D}^s_\mu(\mathcal{T}^n). \qquad (5)$$

Since P and $\bar{\mathcal{Z}}$ are $\mathcal{D}^s_\mu(\mathcal{T}^n)$-right-invariant and C^∞-smooth on $T\mathcal{D}^s_\mu(\mathcal{T}^n)$, it evidently follows from (5) that so is $\mathcal{S}$.

Let $F(t,m)$ be an H^s-vector field (a divergence-free H^s-vector field) on $\mathcal{T}^n$. Denote by $\bar F(t,\eta)$ ($\tilde F(t,\eta)$, respectively) the corresponding right-invariant vector field on $\mathcal{D}^s(\mathcal{T}^n)$ (on $\mathcal{D}^s_\mu(\mathcal{T}^n)$). Consider the equations of second Newton's law

$$\frac{\bar D}{dt}\dot g(t) = \bar F \qquad (6)$$

on $\mathcal{D}^s$ and

$$\frac{\tilde D}{dt}\dot\gamma(t) = \tilde F \qquad (7)$$

on $\mathcal{D}^s_\mu$.

Theorem 1. *Let the vector field $F(t,m)$ on $\mathcal{T}^n$ at any t belong to H^{s+1} and be continuous it t with respect to H^s topology.*

(i) For any vector $v_0\in T_e\mathcal{D}^s(\mathcal{T}^n)$ there exists a unique solution $g(t)$ of (6) with initial conditions $g(0)=e$ and $\dot g(0)=v_0$ for $t\in[0,\varepsilon)$ where $\varepsilon>0$ depends on v_0.

(ii) For any vector $\kappa_0\in T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$ there exists a unique solution of (7) with initial conditions $\gamma(0)=e$ and $\dot\gamma(0)=\kappa_0$ for $t\in[0,\varepsilon)$ where $\varepsilon>0$ depends on κ_0.

The proof (see [5]) is based on the representation the velocity $\dot g(t)$ of solution $g(t)$ of (6) as an integral curve of the vector field

$$\bar{\mathcal{Z}} + \bar F^l \qquad (8)$$

on $T\mathcal{D}^s(\mathcal{T}^n)$ and the velocity $\dot\gamma(t)$ of solution $\gamma(t)$ of (7) as an integral curve of the vector field

$$\mathcal{S} + \tilde F^l. \tag{9}$$

on $T\mathcal{D}^s_\mu(\mathcal{T}^n)$. Those vector fields are at least C^1-smooth since $\bar{\mathcal{Z}}$ and $\mathcal{S}$ are C^∞-smooth while $\bar F^l$ and $\tilde F^l$ are C^1-smooth in g and continuous in t.

Remark 2. It is well-known that for continuous vector fields on infinite-dimensional spaces (manifolds) integral curves may not exist. Nevertheless the above conditions of belonging F to H^{s+1} for any t (that provides C^1-smoothness to $\bar F^l$ and $\tilde F^l$) can be reduced. We refer the reader to [12] where the existence of integral curves on groups of diffeomorphisms was proved for vector fields with Caratheodory type conditions under some additional assumptions that the field is condensing (k-set contraction) with respect to a certain measure of non-compactness.

Consider the vectors $v(t) = TR_{g(t)^{-1}}\dot g(t) \in T_e\mathcal{D}^s(\mathcal{T}^n)$ and $\kappa(t) = TR_{\gamma(t)^{-1}}\dot\gamma(t) \in T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$ obtained by right-hand translations of velocity vectors $\dot g(t) \in T_{g(t)}\mathcal{D}^s(\mathcal{T}^n)$ to solution $g(t)$ of (6) and $\dot\gamma(t)$ to solution $\gamma(t)$ of (7) at any specified t, respectively.

It is shown in [5] (see also [9] and [10]) that $v(t) \in T_e\mathcal{D}^s(\mathcal{T}^n)$ is a vector field on $\mathcal{T}^n$ satisfying the equation of diffuse matter

$$\frac{\partial}{\partial t}v(t,m) + \nabla_{v(t,m)}v(t,m) = 0; \tag{10}$$

and $\kappa(t) \in T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$ is a divergence-free vector field on $\mathcal{T}^n$ satisfying the Euler equation

$$\frac{\partial}{\partial t}\kappa(t,m) + \nabla_{\kappa(t,m)}\kappa(t,m) - \text{grad}p = 0. \tag{11}$$

Thus the curves $g(t)$ and $\gamma(t)$ on $\mathcal{D}^s(\mathcal{T}^n)$ and $\mathcal{D}^s_\mu(\mathcal{T}^n)$, respectively, are the flows of diffuse matter and of perfect incompressible fluid, respectively, on $\mathcal{T}^n$. For $n = 2$ the solutions $\kappa(t)$ of (11) and $\gamma(t)$ of (7) exist for $t \in [0, +\infty)$. This follows from classical results by Kato.

Introduce the operators:

(i) $B : T\mathcal{T}^n \to R^n$, the projection onto the second factor in $T\mathcal{T}^n = \mathcal{T}^n \times R^n$;

(ii) $A(m) : R^n \to T_m\mathcal{T}^n$, the converse to B linear isomorphism from R^n onto the tangent space to $\mathcal{T}^n$ at $m \in \mathcal{T}^n$.

(iii) For $g \in \mathcal{D}^s$ and $m \in \mathcal{T}^n$ consider the isomorphism $Q_g : T_m\mathcal{T}^n \to T_{g(m)}\mathcal{T}^n$ of the form $Q_g = A(g(m)) \circ B$.

Notice that for the natural orthonormal frame b in R^n we have an orthonormal frame $A_m(b)$ in $T_m\mathcal{T}^n$, the field of frames $A(b)$ on $T\mathcal{T}^n$ consists of frames inherited from the constant frame b. Thus for a fixed vector $X \in R^n$ the vector field $A(X)$ on $\mathcal{T}^n$ is constant (has constant coordinates with respect to $A(b)$) and in particular $A(X)$ is C^∞-smooth and divergence-free since such is the constant vector field X on R^n. So, A may be considered as a map $A : R^n \to T_e\mathcal{D}^s_\mu(\mathcal{T}^n) \subset T_e\mathcal{D}^s(\mathcal{T}^n)$.

Consider the map $\bar A : \mathcal{D}^s(\mathcal{T}^n)\times R^n \to T\mathcal{D}^s(\mathcal{T}^n)$ such that $\bar A_e : R^n \to T_e\mathcal{D}^s(\mathcal{T}^n)$ is equal to A, and for every $g \in \mathcal{D}^s(\mathcal{T}^n)$ the map $\bar A_g : R^n \to T_g\mathcal{D}^s(\mathcal{T}^n)$ is obtained from $\bar A_e$ by means of the right-translation:

$$\bar A_g(X) = R_g \circ A_e(X) = (A \circ g)(X). \tag{12}$$

Since A is C^∞-smooth, it follows from ω-lemma that $\bar A$ is C^∞-smooth jointly in $X \in R^n$ and $g \in \mathcal{D}^s(\mathcal{T}^n)$. In particular, the restriction $\bar A : \mathcal{D}^s_\mu(\mathcal{T}^n) \times R^n \to T\mathcal{D}^s_\mu(\mathcal{T}^n)$ is C^∞-smooth and the right-invariant vector field $\bar A(X)$ is C^∞-smooth on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ for every $X \in R^n$.

By the construction, for an arbitrary $f \in \mathcal{D}^s(\mathcal{T}^n)$ and a vector $X \in T_f\mathcal{D}^s(\mathcal{T}^n)$ the vector Q_gX belongs to $T_g\mathcal{D}^s(\mathcal{T}^n)$. In particular, $Q_eX \in T_e\mathcal{D}^s(\mathcal{T}^n)$. Notice that even for $f \in \mathcal{D}^s_\mu(\mathcal{T}^n)$ the operator Q_e does not send $T_f\mathcal{D}^s_\mu$ into $T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$ however $P_eQ_e(T_f\mathcal{D}^s_\mu) = T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$.

Describe the action of Q_g on tangent vectors to $\mathcal{D}^s(\mathcal{T}^n)$ considered as maps and compare it with the right translation TR_g. Recall that $X \in T_e\mathcal{D}^s(\mathcal{T}^n)$, i.e., the vector field on $\mathcal{T}^n$, sends the point $m \in \mathcal{T}^n$ into the vector $(m, X(m))$ where the first component denotes the point in $\mathcal{T}^n$ where the second (vector) component is applied. The right translation TR_g on $T\mathcal{D}^sT^n$ sends the latter into $(g(m), X(g(m)))$ while $Q_g(m, X(m)) = (g(m), X(m))$.

Theorem 3.([11], see also [4]) *$Q_g : T_\eta\mathcal{D}^s(\mathcal{T}^n) \to T_g\mathcal{D}^s(\mathcal{T}^n)$ is the parallel translation in $\mathcal{D}^s(\mathcal{T}^n)$ with respect to $\bar{\mathcal{H}}$.*

Indeed, since the connectors on $\mathcal{T}^n$ and on $\mathcal{D}^s(\mathcal{T}^n)$ coincide, so do the parallel translations.

2. Mean Derivatives

Consider a stochastic process $\xi(t)$, $t \in [0, l]$, defined on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$, taking values in a separable Hilbert space F and such that $\xi(t)$ is an L^1-random variable for all t. The following particular case will play an important role below: $F = R^n$ and

$$\xi(t) = \xi_0 + \int_0^t \beta(s)ds + \sigma w(t) \tag{13}$$

where $w(t)$ is a Wiener process and $\sigma > 0$ is a constant.

Denote by $\mathcal{N}_t^\xi$ the "present" ("now") σ-algebra of $\xi(t)$, i.e., the minimal σ-subalgebra of $\mathcal{F}$ such that $\xi(t)$ at the specified t is measurable with respect to it. We shall suppose that $\mathcal{N}_t^\xi$ is complete (includes all sets of zero probability). Denote the conditional expectation with respect to $\mathcal{N}_t^\xi$ by E_t^ξ. The following notions were introduced by Nelson (see, e.g., [16] – [18]).

Definition 4. *(i) The forward mean derivative $D\xi(t)$ of the process $\xi(t)$ at the moment t is an L^1-random variable of the form*

$$D\xi(t) = \lim_{\Delta t \downarrow 0} E_t^\xi \left(\frac{\xi(t + \Delta t) - \xi(t)}{\Delta t}\right). \tag{14}$$

(ii) The backward mean derivative $D_*\xi(t)$ of $\xi(t)$ at t is an L^1-random variable

$$D_*\xi(t) = \lim_{\Delta t \downarrow 0} E_t^\xi \left(\frac{\xi(t) - \xi(t - \Delta t)}{\Delta t}\right) \tag{15}$$

Here limits are assumed to exist in $L^1(\Omega, \mathcal{F}, \mathsf{P})$ and $\Delta t \downarrow 0$ means that Δt tends to zero from above.

Denote by $\breve{Y}(t, x)$ and $\breve{Y}_*(t, x)$ the measurable vector fields on F such that $D\xi(t) = \breve{Y}(t, \xi(t))$ and $D_*\xi(t) = \breve{Y}_*(t, \xi(t))$, respectively. The existence of those $\breve{Y}(t, x)$ and $\breve{Y}_*(t, x)$, called regressions of $D\xi(t)$ and $D_*\xi(t)$, respectively, with respect to $\xi(t)$, follows from routine facts of Probability Theory (see [19]).

Mean derivatives of Definition 4 are particular cases of the notions determined as follows. Let $x(t)$ and $y(t)$ be L^1-stochastic processes in F defined on $(\Omega, \mathcal{F}, \mathsf{P})$. Introduce y-forward derivative of $x(t)$ by the formula

$$D^y x(t) = \lim_{\Delta t \downarrow 0} E_t^y \left(\frac{x(t + \Delta t) - x(t)}{\Delta t}\right) \tag{16}$$

and y-backward derivative of $x(t)$ by the formula

$$D_*^y x(t) = \lim_{\Delta t \downarrow 0} E_t^y \left(\frac{x(t) - x(t - \Delta t)}{\Delta t}\right) \tag{17}$$

where, of course, the limits are assumed to exist in $L^1(\Omega, \mathcal{F}, \mathsf{P})$.

Remark 5. Let $f : F \to F_1$ be a smooth map. Notice that since the value of mean derivative depends on the "now" σ-algebra of a process, the tangent map Tf sends the derivatives of a process $\eta(t)$ into those of $\xi(t) = f(\eta(t))$ only in the following form: $Tf(D\eta(t)) = D^\eta(\xi(t))$ and $Tf(D^\xi\eta(t)) = D\xi(t)$, but generally speaking $Tf(D\eta(t)) \neq D\xi(t)$. Similar formulae hold for backward derivative: $Tf(D_*\eta(t)) = D_*^\eta(\xi(t))$ and $Tf(D_*^\xi\eta(t)) = D_*\xi(t)$, but generally speaking $Tf(D_*\eta(t)) \neq D_*\xi(t)$.

Lemma 6. (see [9], [10]) *For $t \in (0, 1]$ the equality $D_*w(t) = \frac{w(t)}{t}$ holds.*

Lemma 7. (see [9], [10]) *The integral $\int_0^t \frac{w(s)}{s} ds$ exists almost surely for all $t \in [0, l]$.*

Introduce the process $w_*(t) = -\int_0^t \frac{w(s)}{s} ds + w(t)$.

Lemma 8. (see [9], [10]) $D_*^w w_*(t) = 0$.

Note that the above equality is not valid if D_*^w is replaced by D_*. For example, it is shown in [13] that $w_*^w(t)$ is a Wiener process with respect to its own "past" filtration.

Lemma 9. *For the process*

$$w^*(t) = E_t^w w_*(t). \tag{18}$$

the relation $D_*w^*(t) = 0$ holds.

Proof. By the construction the "now" σ-algebra $\mathcal{N}_t^{w^*}$ of $w^*(t)$ is a σ-subalgebra in $\mathcal{N}_t^w$. Then
$D_*w^*(t) = E_t^{w^*} D_*^w w^*(t) = E_t^{w^*} D_*^w E_t^w w_*(t)$. By Lemma 8.21 of [9] (Lemma 20.5 of [10]) $D_*^w E_t^w w_*(t) = D_*^w w_*(t)$ while the latter is equal to zero by Lemma 8. Q.E.D.

Let $V(t, x)$ be a C^2-smooth vector field on F, and $\xi(t)$ be a stochastic process in F.

Definition 10. *The forward $DV(t, \xi(t))$ and the backward $D_*V(t, \xi(t))$ mean derivatives of V along $\xi(\cdot)$ at t are the L^1-limits of the form*

$$DV(t, \xi(t)) = \\ = \lim_{\Delta t \downarrow 0} E_t^\xi \left(\frac{V(t + \Delta t, \xi(t + \Delta t)) - V(t, \xi(t))}{\Delta t}\right) \tag{19}$$

$$D_*V(t, \xi(t)) =$$

$$= \lim_{\Delta t \downarrow 0} E_t^{\xi}\left(\frac{V(t,\xi(t)) - V(t-\Delta t, \xi(t-\Delta t))}{\Delta t}\right) \quad (20)$$

Consider the regressions of $DV(t,\xi(t))$ and $D_*V(t,\xi(t))$ with respect to $\xi(t)$ denoted by $DV_{\xi(t)}$ and $D_*V_{\xi(t)}$, respectively.

Lemma 11. *For the process (13) in R^n the following formulas hold*

$$DV_{\xi(t)} = \frac{\partial}{\partial t}V + (\check{Y}\cdot\nabla)V + \frac{\sigma^2}{2}\nabla^2 V, \quad (21)$$

$$D_*V_{\xi(t)} = \frac{\partial}{\partial t}V + (\check{Y}_*\cdot\nabla)V - \frac{\sigma^2}{2}\nabla^2 V, \quad (22)$$

where $\nabla = (\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n})$, ∇^2 is the Laplacian, the dot denotes the scalar product in R^n and $\check{Y}(t,X)$ and $\check{Y}_*(t,X)$ are the above-mentioned regressions of $D\xi(t)$ and $D_*\xi(t)$ with respect to $\xi(t)$.

Definition of mean derivatives on manifolds requires some additional constructions involving connections in order to get well-posed (covariant) notions. We refer the reader to [18], [9] and [10] for details. Here we present a variant specially adapted to the groups of diffeomorphisms, a very simple case of the general situation.

For the group $\mathcal{D}^s(\mathcal{T}^n)$ the definitions of mean derivatives by formulae (14) and (15) are well-posed and so need no changes since the operations of addition and subtraction are inherited on this group from $\mathcal{T}^n$ and so from R^n. In order to distinguish the mean derivatives on $\mathcal{D}^s(\mathcal{T}^n)$ from those on $\mathcal{T}^n$ we denote the mean forward (mean backward) derivatives for the former by $\bar{D}$ ($\bar{D}_*$, respectively) and keep the notations D (D_*, respectively) for $\mathcal{T}^n$.

For the process $V(t,\xi(t))$ with values in $T\mathcal{D}^s(\mathcal{T}^n)$ consider $\bar{D}^{\xi}V(t,\xi(t))$ that takes values in $TT\mathcal{D}^s(\mathcal{T}^n)$, tangent to $T\mathcal{D}^s(\mathcal{T}^n)$; $\bar{D}_*^{\xi}(\xi(t), V(t,\xi(t)))$ takes values also in $TT\mathcal{D}^s(\mathcal{T}^n)$. Now the forward $\bar{\mathbf{D}}V(t,\xi(t))$ and the backward $\bar{\mathbf{D}}_*V(t,\xi(t))$ covariant mean derivatives of V along $\xi(\cdot)$ on $\mathcal{D}^s(\mathcal{T}^n)$ at t are defined by the formulae

$$\bar{\mathbf{D}}V(t,\xi(t)) = K \circ \bar{D}^{\xi}V(t,\xi(t)),$$

$$\bar{\mathbf{D}}_*V(t,\xi(t)) = K \circ \bar{D}_*^{\xi}V(t,\xi(t)). \quad (23)$$

On $\mathcal{D}^s_\mu(\mathcal{T}^n)$ the mean derivatives calculated by the above formulae may not be tangent to $\mathcal{D}^s_\mu(\mathcal{T}^n)$. That is why we postulate the following modification.

Definition 12. *(i) The forward mean derivative $\tilde{D}\xi(t)$ of the process $\xi(t)$ on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ at the moment t is an L^1-random variable of the form*

$$\tilde{D}\xi(t) = P \lim_{\Delta t\downarrow 0} E_t^{\xi}\left(\frac{\xi(t+\Delta t) - \xi(t)}{\Delta t}\right). \quad (24)$$

(ii) The backward mean derivative $\tilde{D}_*\xi(t)$ of $\xi(t)$ on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ at t is the L^1-random variable

$$\tilde{D}_*\xi(t) = P \lim_{\Delta t\downarrow 0} E_t^{\xi}\left(\frac{\xi(t) - \xi(t-\Delta t)}{\Delta t}\right). \quad (25)$$

In the same manner we modify the definitions of $D^y x(t)$ and $D_*^y x(t)$ (see (16) and (17)).

Consequently, we introduce the mean derivatives $\tilde{D}^{\xi}V(t,\xi(t)) = TP\bar{D}^{\xi}V(t,\xi(t))$ and $\tilde{D}_*^{\xi}V(t,\xi(t)) = TP\bar{D}_*^{\xi}V(t,\xi(t))$, respectively of the vector field V along $\xi(t)$ on $\mathcal{D}^s_\mu(\mathcal{T}^n)$. The covariant mean derivatives on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ are defined by the formulae

$$\tilde{\mathbf{D}}V(t,\xi(t)) = \tilde{K}\circ\tilde{D}^{\xi}V(t,\xi(t)) = P\circ\bar{\mathbf{D}}V(t,\xi(t)),$$

$$\tilde{\mathbf{D}}_*V(t,\xi(t)) =$$
$$= \tilde{K}\circ\tilde{D}_*^{\xi}V(t,\xi(t)) = P\circ\bar{\mathbf{D}}_*V(t,\xi(t)). \quad (26)$$

3. Description of viscous hydrodynamics

For a point $m \in \mathcal{T}^n$ denote by $exp_m : T_m\mathcal{T}^n \to \mathcal{T}^n$ the map that sends the vector $X \in T_m\mathcal{T}^n$ into the point $m + X$ in $\mathcal{T}^n$. The field of maps exp at all points generates the map $\bar{exp} : T_e\mathcal{D}^s(\mathcal{T}^n) \to \mathcal{D}^s(\mathcal{T}^n)$ that sends the vector $X \in T_e\mathcal{D}^s(\mathcal{T}^n)$ (i.e., a vector field on $\mathcal{T}^n$) into $e + X \in \mathcal{D}^s(\mathcal{T}^n)$ where $e+X$ is the diffeomorphism of $\mathcal{T}^n$ of the form $(e+X)(m) = m + X(m)$.

Consider the composition $\bar{exp}\circ\bar{A}_e : R^n \to \mathcal{D}^s(\mathcal{T}^n)$. By the construction of $\bar{A}_e$ for any $X \in R^n$ we get $\bar{exp}\circ\bar{A}_e(X)(m) = m + X$, i.e., the same vector X is added to any point m. Thus obviously $\bar{exp}\circ\bar{A}_e(X) \in \mathcal{D}^s_\mu(\mathcal{T}^n)$ and so $\bar{exp}\circ\bar{A}_e : R^n \to \mathcal{D}^s_\mu(\mathcal{T}^n)$.

Introduce the process

$$\sigma W^*(t) = \bar{exp}\circ\bar{A}_e(\sigma w^*(t)) \quad (27)$$

in $\mathcal{D}^s(\mathcal{T}^n)$ where $w^*(t)$ is process (18) and $\sigma > 0$ is a real constant. By the construction, for any $\omega \in \Omega$ the corresponding sample trajectory $\sigma W^*_\omega(t)$ is the diffeomorphism of the form $\sigma W^*_\omega(t)(m) = m + \sigma w^*_\omega(t)$ so that the same sample trajectory $\sigma w^*_\omega(t)$ of $\sigma w^*(t)$ is added to each point $m \in \mathcal{T}^n$. In particular, this means that

Lemma 13. *(i) $\sigma W^*(t)$ takes values in $\mathcal{D}^s_\mu(\mathcal{T}^n)$. (ii) at any t specified $\sigma W^*(t)$ may be considered as a random map $\sigma W^*(t) : \mathcal{T}^n \to \mathcal{T}^n$ such that its differential $d\sigma W^*(t) : T_m\mathcal{T}^n \to T_{\sigma W^*(t)m}\mathcal{T}^n$ is equal to I (identity operator); (iii) $\sigma W^*(t)$ may be considered as a stochastic flow on $\mathcal{T}^n$ governed by a certain stochastic differential equation with diffusion term of the form $\sigma w(t)$.*

Specify a vector $u_0 \in T_e\mathcal{D}^s(\mathcal{T}^n)$. Let $g(t)$ be a solution of (6) with $g(0) = e$ and $\dot g(0) = u_0 \in T_e\mathcal{D}^s(\mathcal{T}^n)$. By Theorem 1 $g(t)$ is unique and exists for $t \in [0, \varepsilon)$ where $\varepsilon > 0$ depends on u_0 and it is a flow of diffuse matter on $\mathcal{T}^n$ (see §1).

Consider the stochastic process

$$\eta(t) = \sigma W^*(t) \circ g(t) \tag{28}$$

on $\mathcal{D}^s(\mathcal{T}^n)$. Notice that it is well-posed for $t \in [0, \varepsilon)$ since $g(t)$ exists for such t and $\sigma W^*(t)$ exists for all $t \in [0, +\infty)$ by the construction.

Lemma 14. *The "now" σ-algebra $\mathcal{N}^\eta_t$ of $\eta(t)$ is a σ-subalgebra of $\mathcal{N}^{w^*}_t$, the "now" σ-algebra of process $w^*(t)$.*

Proof. By definition at any t the random variable $\eta(t) = R_{g(t)}\sigma W(t)$ and so its "now" σ-algebra $\mathcal{N}^{\bar\eta}_t$ is the same as for the process $\sigma W(t)$. On the other hand $e\bar xp \circ \bar A_e : R^n \to \mathcal{D}^s(\mathcal{T}^n)$ is a continuous map. Thus (see (27)) $\mathcal{N}^{W^*}_t$ is a σ-subalgebra of $\mathcal{N}^{w^*}_t$. Q.E.D.

Notice that $\mathcal{N}^{w^*}_t$ is a σ-subalgebra of $\mathcal{N}^w_t$ by the construction of $w^*(t)$ (see (18)).

Lemma 15. *$\bar D_*\eta(t) = Q_{\eta(t)}\dot g(t)$.*

Proof. Indeed, $\bar D_*\eta(t) = TR_{g(t)}\bar D^\eta_*\sigma W^*(t) + TL_{\sigma W^*(t)}\bar D^\eta_* g(t) = \sigma TW^*(t) \circ \dot g(t) + \sigma \bar D^\eta_* W^*(t) \circ g(t)$. Obviously $D^\eta_* g(t) = \dot g(t)$ since $g(t)$ is a deterministic process with C^1-trajectory. On the other hand, $\bar D^\eta_* W^*(t) = T(e\bar xp \circ \bar A_e) \circ D^\eta_* w^*(t)$. From Lemma 14 it follows that $D^\eta_* w^*(t) = E^\eta_t D^w w^*(t)$ and $D^w w^*(t)$ is equal to zero by Lemma 9. The assertion of Lemma follows from Lemma 13 and the properties of tangent maps. Q.E.D.

Lemma 16. *$\bar D_*\bar D_*\eta(t) = \bar{\mathcal{Z}}(\bar D_*\eta(t)) + TQ_{\bar D_*\eta(t)}\bar F^l(t, \dot g(t))$.*

Proof. In analogy with the proof of Lemma 15 show that $\bar D_*\bar D_*\eta(t) = \sigma TTW^*(t) \circ \ddot g(t) + \sigma \bar D^\eta_*\bar D^\eta_* W^*(t) \circ g(t)$. Since $\sigma \bar D^\eta_*\bar D^\eta_* W^*(t) = 0$, we get $\bar D_*\bar D_*\eta(t) = \sigma TTW^*(t) \circ \ddot g(t) = TQ_{\bar D_*\eta(t)}\ddot g(t)$. Taking into account (8) and the fact that on the flat torus evidently $TQ_X\mathcal{Z}(Y) = \mathcal{Z}(X)$ we obtain the assertion of Lemma. Q.E.D.

Corollary 17. *$\bar{\mathbf{D}}_*\bar D_*\eta(t) = Q_{\eta(t)}\bar F(t, g(t))$.*

The Corollary follows from Lemma 16 and formula (23). Notice that the above relation is a stochastic analogue of the second Newton's law.

Now introduce the vector $u(t) = ETR^{-1}_{\eta(t)}\bar D_*\eta(t) \in T_e\mathcal{D}^s(\mathcal{T}^n)$. It is an analogue of $v(t)$ in §1. Recall that $u(t)$ is an H^s vector field on $\mathcal{T}^n$ that will be denoted by $u(t, m)$.

Theorem 18. *If the vector field $u(t, m)$ on $\mathcal{T}^n$ is C^1 in t and C^2 in $m \in \mathcal{T}^n$, it satisfies the Burgers equation*

$$\frac{\partial}{\partial t}u(t,m) + < u(t,m), \nabla > u(t,m) - \frac{\sigma^2}{2}\nabla^2 u(t,m) =$$
$$= EF(t, m - \sigma w^*(t)) \tag{29}$$

with initial value u_0.

Proof. One can easily see from the construction that $u(0) = \dot g(0) = u_0$.

Recall that the vector fields $\bar{\mathcal{Z}}$ and $\bar F^l$ are right-invariant. Taking into account formula (9) from [11] and the construction of $\eta(t)$ we get $TR^{-1}_{\eta(t)}Q_{\eta(t)}\bar F(t, g(t)) = Q_e\bar F(t, g(t) \circ \eta(t)^{-1}) = Q_e\bar F(t, \sigma W^*(t)^{-1})$. From this, from Remark 5 and Lemma 16 we obtain

$$\bar D^\eta_* TR^{-1}_{\eta(t)}\bar D_*\eta(t) =$$
$$= \bar{\mathcal{Z}}(TR^{-1}_{\eta(t)}\bar D_*\eta(t)) + (Q_e\bar F(t, \sigma W^*(t)^{-1}))^l. \tag{30}$$

Introduce the process $\eta_t(s) = R^{-1}_{\eta(t)}\eta(s)$. Obviously $\eta_t(t) = e$ and so the conditional expectation $E^{\eta_t}_t$ with respect to its "now" σ-algebra at t is the unconditional expectation E. Thus by Remark 5 $\bar D_*\eta_t(t) = E(TR^{-1}_{\eta(t)}\bar D_*\eta(t)) = u(t)$. From the construction of vector field $\bar{\mathcal{Z}}$ it obviously follows that $E(\bar{\mathcal{Z}}(TR^{-1}_{\eta(t)}\bar D_*\eta(t))) = \bar{\mathcal{Z}}(u(t))$. Then from (30) we get

$$\bar D_*\bar D_*\eta_t(t) =$$
$$= \bar{\mathcal{Z}}(u(t)) + (EQ_e\bar F(t, \sigma W^*(t)^{-1}))^l. \tag{31}$$

Consider the right-invariant vector field $\bar u(t)$ on $\mathcal{D}^s(\mathcal{T}^n)$ generated by $u(t) \in T_e\mathcal{D}^s(\mathcal{T}^n)$. Notice that $\frac{\partial}{\partial t}u(t)$ is a vertical vector at $T_{(e,u(t))}\mathcal{D}^s(\mathcal{T}^n)$ that is the difference between $\bar D_*\bar D_*\eta_t(t)$ and its component tangent to $\bar u(t)$. In order to find this difference recall that by usual machinery the vector $\bar{\mathcal{Z}}(u(t))$ is described as

$$\bar{\mathcal{Z}}(u(t)) = \lim_{\Delta t \downarrow 0} E\left(\frac{u(t) - \bar u(t, \eta_t(t - \Delta t))}{\Delta t}\right)$$

$$-(K \circ \lim_{\Delta t \downarrow 0} E(\frac{u(t) - \bar{u}(t, \eta_t(t - \Delta t)))}{\Delta t})^l$$

where the former vector in the right-hand side is tangent to $\bar{u}(t)$ and the latter is vertical. Then, taking into account (31) we obtain

$$\frac{\partial}{\partial t} u(t) = -(K \circ \lim_{\Delta t \downarrow 0} E(\frac{u(t) - \bar{u}(t, \eta_t(t - \Delta t))}{\Delta t}))^l +$$

$$+(EQ_e \bar{F}(t, \sigma W^*(t)^{-1}))^l.$$

All vectors in the last expression are vertical, i.e., tangent to the linear space $T_e \mathcal{D}^s(\mathcal{T}^n)$. As usual for the theory of differential equations in linear space we can consider the vectors, tangent to linear space, as belonging to it. Then the last expression takes the form

$$\frac{\partial}{\partial t} u(t) = -K \circ \lim_{\Delta t \downarrow 0} E(\frac{u(t) - \bar{u}(t, \eta_t(t - \Delta t))}{\Delta t}) +$$

$$+EQ_e \bar{F}(t, \sigma W^*(t)^{-1}). \qquad (32)$$

It is easy to see that the process $\eta_t(s)$ on $\mathcal{D}^s(\mathcal{T}^n)$ can be considered as a stochastic flow on $\mathcal{T}^n$ whose backward mean derivative at $s = t$ is $u(t, m)$ and the diffusion term is $\sigma w(t)$. So we can differentiate $u(t, m)$ along this process by formula (20) and apply formula (22) to obtain

$$-K \circ \lim_{\Delta t \downarrow 0} E(\frac{u(t) - \bar{u}(t, \eta_t(t - \Delta t))}{\Delta t})(m) =$$

$$-(< u(t, m), \nabla > u(t, m) - \frac{\sigma^2}{2} \nabla^2 u(t, m)).$$

Thus (32) transforms into (29). Q.E.D.

Now let us turn to stochastic perturbations of solutions of (7). Using analogous scheme of arguments we shall obtain solutions of the Navier-Stokes equation. The only serious modification involves the projector P into various formulae in order not to leave the manifold $\mathcal{D}^s_\mu(\mathcal{T}^n)$ and so we describe this material more briefly.

Specify a vector $U_0 \in T_e \mathcal{D}^s_\mu(\mathcal{T}^n)$. Let $\gamma(t)$ be a solution of (7) with $\gamma(0) = e$ and $\dot{\gamma}(0) = U_0 \in T_e \mathcal{D}^s_\mu(\mathcal{T}^n)$. By Theorem 1 $\gamma(t)$ is unique and exists for $t \in [0, \varepsilon)$ where $\varepsilon > 0$ depends on U_0 and it is a flow of perfect incompressible on $\mathcal{T}^n$ (see §1).

Consider the stochastic process

$$\xi(t) = \sigma W^*(t) \circ \gamma(t) \qquad (33)$$

on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ (recall that $W^*(t)$ in fact takes values in $\mathcal{D}^s_\mu(\mathcal{T}^n)$, see above). Notice that it is well-posed for $t \in [0, \varepsilon)$ since $\gamma(t)$ exists for such t and $\sigma W^*(t)$ exists for all $t \in [0, +\infty)$ by the construction. Lemma 14 remains true for $\xi(t)$. The analogues of Lemmas 15 and 16 and of Corollary 17 take the form:

Lemma 19. $\tilde{D}_* \xi(t) = PQ_{\xi(t)} \dot{\gamma}(t)$.

Lemma 20.
$\tilde{D}_* \tilde{D}_* \xi(t) = \mathcal{S}(\tilde{D}_* \xi(t)) + (PQ_{\tilde{D}_* \xi(t)} \tilde{F}(t, \dot{\gamma}(t)))^l$.

Corollary 21. $\tilde{\mathbf{D}}_* \tilde{D}_* \xi(t) = PQ_{\xi(t)} \tilde{F}(t, \gamma(t))$.

Corollary 21 means that $\xi(t)$ satisfies a stochastic analogue of the second Newton's law on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ (cf. Corollary 17).

Now introduce the vector $U(t) = ETR^{-1}_{\xi(t)} \tilde{D}_* \xi(t) \in T_e \mathcal{D}^s_\mu(\mathcal{T}^n)$. It is an analogue of $\kappa(t)$ in §1. Recall that $U(t)$ is an H^s divergence-free vector field on $\mathcal{T}^n$ that will be denoted by $U(t, m)$.

Theorem 22. *If the vector field $U(t, m)$ on $\mathcal{T}^n$ is C^1 in t and C^2 in $m \in \mathcal{T}^n$, it satisfies the Navier-Stokes equation*

$$\frac{\partial}{\partial t} U(t, m) + < U(t, m), \nabla >$$

$$> U(t, m) - \frac{\sigma^2}{2} \nabla^2 U(t, m) - \mathrm{grad} p =$$

$$= EPF(t, m - \sigma w^*(t)) \qquad (34)$$

with initial value U_0.

Proof. The scheme of arguments here is quite analogous to the proof of Theorem 18 and so we clarify only the points where the main modifications arise.

One can easily see from the construction that $U(0) = \dot{\gamma}(0) = U_0$.

The analogue of (30) takes the form

$$\tilde{D}^\xi_* TR^{-1}_{\xi(t)} \tilde{D}_* \xi(t) = \mathcal{S}(TR^{-1}_{\xi(t)} \tilde{D}_* \xi(t)) +$$

$$+(PQ_e \tilde{F}(t, \sigma W^*(t)^{-1}))^l. \qquad (35)$$

Introduce $\xi_t(s) = R^{-1}_{\xi(t)} \xi(s)$, the analogue of $\eta_t(s)$ having the same properties and in particular satisfying the following analogue of (31):

$$\tilde{D}_* \tilde{D}_* \eta_t(t) =$$

$$= \mathcal{S}(U(t)) + (EPQ_e \tilde{F}(t, \sigma W^*(t)^{-1}))^l. \qquad (36)$$

As well as above $\frac{\partial}{\partial t} U(t)$ is a vertical vector at $T_{(e, U(t))} \mathcal{D}^s_\mu(\mathcal{T}^n)$ that is the difference between $\tilde{D}_* \tilde{D}_* \xi_t(t)$ and its component tangent to $\bar{U}(t)$, the right-invariant vector field on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ generated by $U(t) \in T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$. Here this difference can be found from the equality

$$\mathcal{S}(U(t)) = \lim_{\Delta t \downarrow 0} E(\frac{U(t) - \bar{U}(t, \tilde{\xi}_t(t - \Delta t))}{\Delta t})$$

$$-P \circ K \circ \lim_{\Delta t \downarrow 0} E(\frac{U(t) - \bar{U}(t, \tilde{\xi}_t(t - \Delta t))}{\Delta t})$$

(recall that $P \circ K = \tilde{K}$) where the former vector in the right-hand side is tangent to $\bar{U}(t)$ and the latter is vertical. Then, we obtain from (36)

$$\frac{\partial}{\partial t} U(t) = -P \circ K \circ \lim_{\Delta t \downarrow 0} E(\frac{U(t) - \bar{U}(t, \tilde{\xi}_t(t - \Delta t))}{\Delta t}) +$$

$$+ EPF(t, \sigma W^*(t)^{-1}). \qquad (37)$$

This is the analogue of (32), i.e., it is written in the linear space $T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$.

The process $\tilde{\xi}_t(s)$ on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ can be considered as a stochastic flow on $\mathcal{T}^n$ whose backward mean derivative at $s = t$ is $U(t, m)$. In order to differentiate U along $\xi_t(s)$ by formula (25) we may differentiate $U(t, m)$ along this flow by (20) and apply P to the result obtained by (22). Taking into account (3) we get

$$-P \circ K \circ \lim_{\Delta t \downarrow 0} E(\frac{U(t) - \bar{U}(t, \tilde{\xi}_t(t - \Delta t))}{\Delta t}) =$$

$$-P(< U(t, m), \nabla > U(t, m) - \frac{\sigma^2}{2} \nabla^2 U(t, m)) =$$

$$-(< U(t, m), \nabla > U(t, m) - \frac{\sigma^2}{2} \nabla^2 U(t, m) - \operatorname{grad} p).$$

so that (37) transforms into (34). Q.E.D.

Notice that the processes $\eta(t)$ and $\xi(t)$ are constructed for $s > \frac{n}{2} + 1$ from existing solutions of (6) and (7), respectively, for $F(t, m)$ belonging to H^{s+1} at any t and continuous in t with respect to H^s topology. Thus the vector fields $u(t, m)$ and $U(t)$ from Theorems 18 and 22, respectively, do exist but may not be smooth enough to satisfy equations (29) and (34), respectively, in classical sense.

Definition 23. *If the vector field $u(t, m)$ or $U(t, m)$ on $\mathcal{T}^n$ is not smooth enough to satisfy (29) or (34), respectively, in classical sense, we say that it is a generalized solution of the corresponding equation.*

If $s > \frac{n}{2} + 2$, the generalized solutions are classical ones since they have enough classical derivatives. Their flows describe the motion of corresponding viscous fluids on $\mathcal{T}^n$. These flows may be called "expectations" of the stochastic processes $\eta(t)$ and $\xi(t)$, respectively, lying under the motion of viscous fluids. It should be pointed out that the processes and their "expectations" (flows of the generalized solutions) do exist for $s > \frac{n}{2} + 1$.

Let us summarize what we can say about (generalized) solutions of Burgers and Navier-Stokes equations in the case under consideration. Denote $EF(t, m - \sigma w^*(t))$ by $\Phi(t, m)$. Notice that if the force F does not depend on m, $\Phi(t) = F(t)$. In particular, $\Phi = 0$ if $F = 0$.

A. Navier-Stokes Equation.

1. *Let $s > \frac{1}{2}n + 1$, U_0 be an H^s divergence free vector field on $\mathcal{T}^n$, $\sigma > 0$ be a real number and $F(t, m)$ be a divergence-free vector field on $\mathcal{T}^n$ belonging to H^{s+1} at any t and continuous in t with respect to topology H^s. Then there exists a generalized solution $U(t)$ of Navier-Stokes equation with viscousity $\frac{\sigma^2}{2}$ and external force $P\Phi(t, m)$, having initial value U_0 and well-posed on the same time interval $[0, \varepsilon)$ as the solution $\kappa(t)$, $\kappa(0) = U_0$ of Euler equation with force $F(t, m)$ where $\varepsilon > 0$ depends on U_0.*

2. *$U(t)$ tends to the solution $\kappa(t)$ as $\sigma \to 0$, (i.e., the viscousity coefficient tends to zero).*

3. *For $n = 2$ the generalized solution of Navier-Stokes equation exists for all $t \in [0, +\infty)$ since solutions of Euler equation exist for those t (see §1).*

4. *For $s > \frac{1}{2}n + 2$ the above-mentioned generalized solution of Navier-Stokes equation is a classical solution.*

B. Burgers equation.

5. *Let $s > \frac{1}{2}n + 1$, u_0 be an H^s vector field on $\mathcal{T}^n$, $\sigma > 0$ be a real number and $F(t, m)$ be a vector field on $\mathcal{T}^n$ belonging to H^{s+1} at any t and continuous in t with respect to topology H^s. Then there exists a generalized solution $u(t)$ of Burgers equation with viscousity $\frac{\sigma^2}{2}$ and external force $\Phi(t, m)$, having initial value u_0 and well posed on the same time interval $[0, \varepsilon)$ as the solution $v(t)$, $v(0) = u_0$ of diffuse matter equation (6) with external force $F(t, m)$, where $\varepsilon > 0$ depends on u_0.*

6. *$u(t)$ tends to $v(t)$ as $\sigma \to 0$, (i.e., the viscousity coefficient tends to zero).*

7. *For $s > \frac{1}{2}n + 2$ the above-mentioned generalized solution of Burgers equation is a classical solution.*

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