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BOUNDARY FUNCTIONS METHOD FOR NONLINEAR SINGULARLY PERTURBED TIME DELAY SYSTEMS

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The paper deals with singularly perturbed differential equations with time-delay h. For Cauchy problem asymptotical expansion of power order ε^{N+1} uniformly bounded with respect to a small parameter ε and $t \in [-h,T]$ is constructed. The algorithm presented coincides with the known boundary functions method by A.B.Vasil'eva for ordinary differential equations.

1. Introduction

As it is well known, the singularly perturbed systems arise in the problems of chemical kinetics, biology, mechanics et. al. Singular perturbations legitimize simplifications of dynamic models. One of them is to neglegt some of "small" time constants, masses, capacities and similar «parasitic» parameters, which increase the dynamic order of the model. At the same time such systems often contain a time-delay that can be caused with measuring and executive devices.

First results regarding singularly perturbed equations were obtained by A.N. Tikhonov [1], A.B. Vasil'eva [2], [3], R.E. O'Malley [4]. The important applications of this theory for the control systems are tdiscussed in the known overview by P.V. Kokotovic [5].

This paper is essentially based on the boundary functions method by A.B. Vasil'eva that allows to get approximative solutions in the form of the finite series of regular and boundary functions.

We consider Cauchy problem for a delay system

$$\frac{dx}{dt} = f(x, x(t-h), z), \tag{1.1}$$

$$\varepsilon\frac{dz}{dt} = g(x, x(t-h), z), \tag{1.2}$$

where ε is a small positive parameter. If one start to construct an expansion of regular solution of (1.1), (1.2) in the common form

$$x = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \dots,$$

$$z = z_0(t) + \varepsilon z_1(t) + \varepsilon^2 z_2(t) + \dots,$$

then $z_i(t) (i = 1,2,\dots)$ can have jumps in the points $t = h, 2h, \dots, ih$. «To stitch up» these jumps, it is necessary to add to $z_i(t)$ the new jump boundary functions $(s = 1,2,\dots,i)$,

$$\tau_s = \begin{cases} t - sh, & t \le sh, \\ (t-sh)/\varepsilon, & t > sh. \end{cases} \tag{1.3}$$

To neutralize the jumps caused by Π_{is} in the first equation of (1.1) it is necessary to add to $x_i(t)$ the new boundary functions π_{is}.

Thus we conclude that asymptotical expansions for time-delay system (1.1), (1.2) should contain both the regular series and the series of boundary functions located in the right-hand neighborhoods of the points $t = h, 2h, \dots$. The way to look for the terms for these expansions is developed in Section 3. Further, in Section 4 the estimates for remainder terms are given. These results can be applied for studying of rocket motors control [9], for chemical reactors analysis [11] et al.

Notations and Problem Formulation

Let R^n and R^m be n-dimensional and m-dimensional real spaces respectively, $\Omega_0 \subset R^n$, $\Omega \subset R^m$ be open bounded domains. By $C^k(G, R^l)$ we denote a space of k-times continuously differentiable functions $f: G \to R^l$. Let N be an integer positive number, $h > 0$, $T > 0$ with $Nh < T$. At last, let

$$f \in C^{N+2}(\Omega_0 \times \Omega_0 \times \Omega, R^n)$$
$$g \in C^{N+2}(\Omega_0 \times \Omega_0 \times \Omega, R^m),$$
$$\theta \in C^{N+2}([-h,0], R^n).$$

Consider an initial Cauchy problem

$$\frac{dx}{dt} = f(x(t), x(t-h), z(t)), \tag{2.1}$$

$$\varepsilon \frac{dz}{dt} = g(x(t), x(t-h), z(t), t \in (0,T]), \tag{2.2}$$

$$x(t) = \theta(t)(-h \le t \le 0), \tag{2.3}$$

$$z(0) = \alpha \in R^m, \tag{2.4}$$

where ε is a small positive parameter. By solution $x(t)$, $z(t)$ of the system (2.1), (2.2) we mean absolutely continuous functions, which satisfy (2.1), (2.2) for $t \in (0,T], t \ne h, 2h, \ldots$.

Now we introduce the new fast times $\tau_s (s = 0,1,\ldots,N+2)$ by (1.3). Obviously, $\tau_s(t-h) = \tau_{s+1}$. As usually, by $\pi_{ks}(\tau_s)(s = 0,1,2,\ldots,N+2)$ we denote continuous boundary functions with

$$\|\pi_{ks}(\tau_s)\| \le C exp(-\mu\tau_s), \ \ \tau_s \ge 0, \ \ C > 0, \ \ \mu > 0.$$

Let $\pi_{ks}(\tau_s) = \pi_{ks}(0) = const$ for $\tau_s \le 0$ and

$$\bar{\pi}_{ks}(\tau_s) = \begin{cases} \pi_{ks}(\tau_s), & t \le sh, \\ 0, & t > sh. \end{cases}$$

The function $\bar{\pi}_{ks}$ is discontinuous in the point $\tau_s = 0$. It is a regular part of the boundary layer π_{ks}.

By $\Pi_{ks}(\tau_s)(s = 1,2,\ldots,N+2)$ we denote generally speaking a discontinuous boundary function with

$$\|\Pi_{ks}(\tau_s)\| \le C exp(-\mu\tau_s), \ \ C > 0, \ \ \tau_s \ge 0, \ \ \mu > 0,$$
$$\Pi_{ks}(\tau_s) = 0, \ \ \tau_s < 0.$$

Let $\Pi_{k0}(\tau_0)$ be a usual boundary layer of τ_0.

An approximate solution of the initial problem (2.1), (2.2) we will look for in the form

$$x_* = x_0(t) + \varepsilon[x_1(t) + \pi_{10}(\tau_0)] +$$
$$\varepsilon^2[x_2(t) + \pi_{20}(\tau_0) + \pi_{21}(\tau_1)] + \ldots +$$
$$+ \varepsilon^N[x_N(t) + \pi_{N0}(\tau_0) + \ldots + \pi_{NN}(\tau_N)] +$$
$$+ \varepsilon^{N+1}[\pi_{N+1,0}(\tau_0) + \ldots + \pi_{N+1,N}(\tau_N)], \tag{2.5}$$

$$z_* = z_0(t) + \Pi_{00}(\tau_0) + \varepsilon[z_1(t) + \Pi_{10}(\tau_0) + \Pi_{11}(\tau_1)] +$$
$$+ \ldots + \varepsilon^N[z_N(t) + \Pi_{N0}(\tau_0) + \ldots + \Pi_{NN}(\tau_N)], \tag{2.6}$$

where $x_i, \pi_{is} \in R^n$, $z_i, \Pi_{is} \in R^m$.

We will see below that functions $z_i(t)(i \ge 1)$ can have jumps in the points $t = h, 2h, \ldots, ih$. To eliminate these jumps one must use the discontinuous boundary functions $\Pi_{is}(\tau_s)(s \ge 1)$. Thus, the functions

$$z_i(t) + \Pi_{i0}(\tau_0) + \Pi_{i1}(\tau_1) + \ldots + \Pi_{ii}(\tau_i) \ \ (i = 1,2,\ldots,N)$$

will be all the same an absolutely continuous. Our nearest goal is to indicate a simple way to construct the expansions (2.5), (2.6).

Seeking of Expansions (2.5), (2.6)

For brevity, in the next part of the study we shall use the notations below:

$$A = A(t,\tau_0) = f_x(x_0(t), x_0(t-h), z_0(t) + \Pi_{00}(\tau_0)),$$
$$B = B(t,\tau_0) = f_y(x_0(t), x_0(t-h), z_0(t) + \Pi_{00}(\tau_0)),$$
$$L = L(t,\tau_0) = f_z(x_0(t), x_0(t-h), z_0(t) + \Pi_{00}(\tau_0)),$$
$$C = C(t,\tau_0) = g_x(x_0(t), x_0(t-h), z_0(t) + \Pi_{00}(\tau_0)),$$
$$D = D(t,\tau_0) = g_y(x_0(t), x_0(t-h), z_0(t) + \Pi_{00}(\tau_0)),$$
$$M = M(t,\tau_0) = g_z(x_0(t), x_0(t-h), z_0(t) + \Pi_{00}(\tau_0)),$$

where f_x, g_x are the first Frech\'et derivatives of f, g with respect to x, and f_y, g_y are the first Frechét derivatives with respect to second argument y, and, finally, f_z, g_z are the first Frech\'et derivative of f, g at z. For brevity, we shall omit the arguments t, τ at the functions $x_i(t), z_i(t), \pi_{is}(\tau_s), \Pi_i(\tau_s)$. Due to time-delay h in (2.1), (2.2) we need to compute these functions in the points $t-h, t-2h, \ldots$. In these cases the arguments of the functions $x_i, z_i, \pi_{is}, \Pi_i$ should be exactly written.

Now substituting formally x_*, z_* $ into equations (2.1), (2.2) and using Taylor series, we get below follow formulas from page 8

$$\frac{dx_0}{dt}+\varepsilon\left(\frac{dx_1}{dt}+\varepsilon^{-1}\frac{d\pi_{10}}{d\tau_0}\right)+\varepsilon^2\left(\frac{dx_2}{dt}+\varepsilon^{-1}\frac{d\pi_{20}}{d\tau_0}+\varepsilon^{-1}\frac{d\pi_{21}}{d\tau_1}\right)+\ldots=$$
$$= f(x_0+\varepsilon(x_1+\pi_{10})+\varepsilon^2(x_2+\pi_{20}+\pi_{21})+\ldots$$
$$+x_0(t-h)+\varepsilon(x_1(t-h)+\pi_{10}(\tau_1)+\varepsilon^2(x_2(t-h)+\pi_{20}(\tau_1)+\pi_{21}(\tau_2))+\ldots,$$
$$z_0+\Pi_{00}+\varepsilon(z_1+\Pi_{10}+\Pi_{11})+\ldots)=$$
$$= f(x_0,x_0(t-h),z_0+\Pi_{00})+$$
$$+\varepsilon A(x_1+\pi_{10}+\varepsilon(x_2+\pi_{20}+\pi_{21})+\ldots)+$$
$$+\varepsilon B(x_1(t-h)+\pi_{10}(\tau_1)+\varepsilon(x_2(t-h)+\pi_{20}(\tau_1)+\pi_{21}(\tau_2))+\ldots)+$$
$$+\varepsilon L(z_1+\Pi_{10}+\Pi_{11}+\varepsilon(z_2+\Pi_{20}+\Pi_{21}+\Pi_{22})+\ldots+$$
$$+\frac{\varepsilon^2}{2}f_{xx}(x_0,x_0(t-h),z_0+\Pi_{00})(x_1+\pi_{10})(x_1+\pi_{10})+$$
$$+\frac{\varepsilon^2}{2}f_{yy}(x_0,x_0(t-h),z_0+\Pi_{00})(x_1(t-h)+\pi_{10}(\tau_1))(x_1(t-h)+\pi_{10}(\tau_1))+$$
$$+\frac{\varepsilon^2}{2}f_{zz}(x_0,x_0(t-h),z_0+\Pi_{00})(z_1+\Pi_{10}+\Pi_{11})(z_1+\Pi_{10}+\Pi_{11})+$$
$$+\varepsilon f_{xy}(x_0,x_0(t-h),z_0+\Pi_{00})(x_1+\pi_{10})(x_1(t-h)+\pi_{10}(\tau_1))+$$
$$+\varepsilon f_{xz}(x_0,x_0(t-h),z_0+\Pi_{00})(x_1+\pi_{10})(z_1+\Pi_{10}+\Pi_{11})+$$
$$+\varepsilon f_{yz}(x_0,x_0(t-h),z_0+\Pi_{00})(x_1(t-h)+\pi_{10}(\tau_1))(z_1+\Pi_{10}+\Pi_{11})+\ldots \tag{3.1}$$

Similarly,

$$\varepsilon(\frac{dz_0}{dt}+\varepsilon^{-1}\frac{d\Pi_{00}}{d\tau_0}+\varepsilon(\frac{dz_1}{dt}+\varepsilon^{-1}\frac{d\Pi_{10}}{d\tau_0}+\varepsilon^{-1}\frac{d\Pi_{11}}{d\tau_1})+$$
$$+\varepsilon\frac{d\Pi_{20}}{d\tau_0}+\varepsilon\frac{d\Pi_{21}}{d\tau_1}+\varepsilon\frac{d\Pi_{22}}{d\tau_2}+\ldots)=$$
$$g(x_0,x_0(t-h),z_0+\Pi_{00})+$$
$$+\varepsilon C(x_1+\pi_{10}+\varepsilon(x_2+\pi_{20}+\pi_{21})+\ldots)+$$
$$+\varepsilon D(x_1(t-h)+\pi_{10}(\tau_1)+\varepsilon(x_2(t-h)+\pi_{20}(\tau_1)+\pi_{21}(\tau_2))+\ldots)+$$
$$+\varepsilon M(z_1+\Pi_{10}+\Pi_{11}+\varepsilon(z_2+\Pi_{20}+\Pi_{21}+\Pi_{22})+\ldots+$$
$$+\frac{\varepsilon^2}{2}g_{xx}(x_0,x_0(t-h),z_0+\Pi_{00})(x_1+\pi_{10})+$$
$$+\frac{\varepsilon^2}{2}g_{yy}(x_0,x_0(t-h),z_0+\Pi_{00})(x_1(t-h)+\pi_{10}(\tau_1))(x_1(t-h)+\pi_{10}(\tau_1))+$$
$$+\frac{\varepsilon^2}{2}g_{zz}(x_0,x_0(t-h),z_0+\Pi_{00})(z_1+\Pi_{10}+\Pi_{11})(z_1+\Pi_{10}+\Pi_{11})+$$
$$+\varepsilon^2 g_{xy}(x_0,x_0(t-h),z_0+\Pi_{00})(x_1+\pi_{10})(x_1(t-h)+\pi_{10}(\tau_1))+$$
$$+\varepsilon^2 g_{xz}(x_0,x_0(t-h),z_0+\Pi_{00})(x_1+\pi_{10})(z_1+\Pi_{10}+\Pi_{11})+$$
$$+\varepsilon^2 g_{yz}(x_0,x_0(t-h),z_0+\Pi_{00})(x_1(t-h)+\pi_{10}(\tau_1))(z_1+\Pi_{10}+\Pi_{11})+\ldots \tag{3.2}$$

Here f_{xx}, ... , g_{yz} are the second order Frechét derivatives.

Our distant purpose is to transform the right-hand parts of (3.1), (3.2) in the sum of regular series and boundary functions series as (2.5), (2.6). However, implementing it at once is very difficult. Therefore we shall do it step-by-step. First, we shall separate the terms of power order ε^0 and transform them to form (2.5), (2.6), simultaneously seeking x_0, z_0, Π_{00} and π_{10}.

I. Consider $f(x_0(t),x_0(t-h),z_0(t)+\Pi_{00}(\tau_0))$, and using the Vasil'eva approach [2] transform it as follows

$$f(x_0(t),x_0(t-h),z_0(t)+\Pi_{00}(\tau_0))=f(x_0(t),x_0(t-h),z_0(t))+$$
$$+(f(x_0(t),x_0(t-h),z_0(t)+\Pi_{00}(\tau_0))-f(x_0(t),x_0(t-h),z_0(t))=$$
$$= f(x_0(t),x_0(t-h),z_0(t))+$$
$$+(f(x_0(\varepsilon\tau_0),x_0(\varepsilon\tau_0-h),z_0(\varepsilon\tau_0)+\Pi_{00}(\tau_0))-$$
$$-f(x_0(\varepsilon\tau_0),x_0(\varepsilon\tau_0-h),z_0(\varepsilon\tau_0)).$$

Expanding the last term in the Taylor series of power ε^i we obtain

$$f(x_0(t),x_0(t-h),z_0(t)+\Pi_{00}(\tau_0))=f(x_0(t),x_0(t-h),z_0(t))+$$
$$+\Pi_{00}f+\varepsilon\Pi_{01}f+\varepsilon^2\Pi_{02}f+\ldots, \tag{3.3}$$

where $\Pi_{00}f,\Pi_{01}f,\Pi_{02}f$ are the boundary functions and, for example,

$$\Pi_{00}f=f(x_0(0),x_0(-h),z_0(0)+\Pi_{00}(\tau_0))-f(x_0(0),x_0(-h),z_0(0)). \tag{3.4}$$

In the similar way it is easy to get

$$g(x_0(t),x_0(t-h),z_0(t)+\Pi_{00}(\tau_0))=$$
$$=g(x_0(t),x_0(t-h),z_0(t))+\Pi_{00}g+\varepsilon\Pi_{10}g+\varepsilon^2\Pi_{20}g+\ldots, \tag{3.5}$$

where

$$\Pi_{00}g = g(x_0(0),x_0(-h),z_0(0)+\Pi_{00}(\tau_0)) - g(x_0(0),x_0(-h),z_0(0)) \quad (3.6)$$

Therefore, it seems naturally to determine the regular terms x_0, z_0 in the following manner:

$$\frac{dx_0}{dt} = f(x_0,x_0(t-h),z_0), \quad t\in(0,T], \quad (3.7)$$

$$g(x_0,x_0(t-h),z_0)=0, \quad (3.8)$$

$$x_0(t)=\theta(t) \quad (-h\le t\le 0). \quad (3.9)$$

Assumption 1. For all $x,y\in\Omega_0$ the equation

$$g(x,y,z)=0$$

has a unique solution

$$z=\varphi(x,y)\in C^{N+2}(\Omega_0\times\Omega_0,R^m),$$

moreover, the matrix

$$\mathcal{M}(x,y)=g_z(x,y,\varphi(x,y))$$

is Gurvitz for all $x,y\in\Omega_0$, i.e. for some $\mu>0$ and all eigenvalues $\lambda_i(x,y)$ of $\mathcal{M}$ the estimate

$$\max_i \sup_{x,y\in\Omega_0} \operatorname{Re}\lambda_i(x,y)\le-\mu \quad (3.10)$$

hold.

Then the equation (3.8) is equivalent to

$$z_0=\varphi(x_0(t),x_0(t-h)) \quad (0\le t\le T), \quad (3.11)$$

and for x_0 we have the initial Cauchy problem

$$\frac{dx_0}{dt}=f_0(x_0,x_0(t-h)), \quad t\in(0,T]. \quad (3.12)$$

$$x_0(t)=\theta(t), \quad t\in[-h,0], \quad (3.13)$$

where

$$f_0(x_0,x_0(t-h),\varphi x_0,x_0(t-h))$$

Assumption 2. The initial problem (3.12), (3.13) has a unique solution

$$x_0(t)\in\Omega_0 \quad (-h\le t\le T), \quad (3.14)$$

moreover, some σ - neighborhood of $x_0(t)$ belong to $\Omega_0\subset R^n$.

Remark 3.1. The functions $\dot{x}_0(t-h)$ and $\dot{z}_0(t)$ can have the jumps at the point $t=h$.

We define the boundary function Π_{00} as a solution of Cauchy problem

$$\frac{dz}{d\tau_0}=g(x_0(0),x_0(-h),z_0(0)+z)-g(x_0(0),x_0(-h),z_0(0)), \quad (3.15)$$

$$z(0)=\alpha-z_0(0). \quad (3.16)$$

Obviously, the equation (3.15) has a trivial steady state $z=0$.

Assumption 3. The solution z of initial problem (3.15), (3.16) tends to zero as $\tau_0\to\infty$.

Remark 3.2. It is easy to show the function z is boundary function, i.e. for any $C_0>0$, $\mu>0$ we have

$$\|z(\tau_0)\|\le C\exp(-\mu\tau_0), \quad \tau_0>0. \quad (3.17)$$

That is why we denote $z(\tau_0)$ by $\Pi_{00}(\tau_0)$.

Assumption 4. A curve

$$w=z_0(\varepsilon\tau_0)+\Pi_{00}(\tau_0) \quad (0<\tau_0\le T/\varepsilon)$$

and some of its σ - neighborhood belong to $\Omega\subset R^m$.

To end, put

$$\pi_{10}(\tau_0)=-\int_{\tau_0}^{\infty}[f(x_0(0),x_0(-h),z_0(0)+\Pi_{00}(s))- - f(x_0(0),x_0(-h),z_0(0))]ds \quad (\tau_0>0), \quad (3.18)$$

$$\pi_{10}(\tau_0)=\pi_{10}(0), \quad \tau_0\le 0.$$

II. As above, transform the matrix coefficients A,B,C,D,L,M, using the Vasil'eva approach. So,

$$A=f_x(x_0,x_0(t-h),z_0+\Pi_{00})=f_x(x_0,x_0(t-h),z_0)+$$
$$+(f_x(x_0(\varepsilon\tau_0),x_0(\varepsilon\tau_0-h),z_0(\varepsilon\tau_0)+\Pi_{00}(\tau_0)),$$
$$-f_x(x_0(\varepsilon\tau_0),x_0(\varepsilon\tau_0-h),z_0(\varepsilon\tau_0)))=$$

$$= A_0(t) + \pi_{00}f_x + \varepsilon\pi_{10}f_x + \varepsilon^2\pi_{20}f_x + ..., \quad (3.19)$$

where

$$A_0(t) = f_x(x_0, x_0(t-h), z_0),$$
$$\pi_{00}f_x = f_x(x_0(0), x_0(-h), z_0(0) + \Pi_{00}(\tau_0)) - f_x(x_0(0), x_0(-h), z_0(0)),$$
....

Analogously,

$$B = B_0(t) + \pi_{00}f_y + \varepsilon\pi_{10}f_y + \varepsilon^2\pi_{20}f_y + ...,$$
$$B_0(t) = f_y(x_0, x_0(t-h), z_0), \quad (3.20)$$
$$\pi_{00}f_y = f_y(x_0(0), x_0(-h), z_0(0) + \Pi_{00}(\tau_0)) - f_y(x_0(0), x_0(-h), z_0(0)),$$
....

$$C = C_0(t) + \pi_{00}g_x + \varepsilon\pi_{10}g_x + \varepsilon^2\pi_{20}g_x + ..., \quad (3.21)$$
$$C_0(t) = g_x(x_0, x_0(t-h), z_0),$$
$$\pi_{00}g_x = f_x(x_0(0), x_0(-h), z_0(0) + \Pi_{00}(\tau_0)) - g_x(x_0(0), x_0(-h), z_0(0)),$$
$$D = D_0(t) + \pi_{00}g_y + \varepsilon\pi_{10}g_y + \varepsilon^2\pi_{20}g_y + ..., \quad (3.22)$$
$$D_0(t) = g_y(x_0, x_0(t-h), z_0),$$
$$\pi_{00}f_y = g_y(x_0(0), x_0(-h), z_0(0) + \Pi_{00}(\tau_0)) - g_y(x_0(0), x_0(-h), z_0(0)),$$
....

$$L = L_0(t) + \pi_{00}f_z + \varepsilon\pi_{10}f_z + \varepsilon^2\pi_{20}f_z + ..., \quad (3.23)$$
$$L_0(t) = f_z(x_0, x_0(t-h), z_0),$$
$$\pi_{00}f_z = f_z(x_0(0), x_0(-h), z_0(0) + \Pi_{00}(\tau_0)) - f_z(x_0(0), x_0(-h), z_0(0)),$$
....

$$M = M_0(t) + \pi_{00}g_z + \varepsilon\pi_{10}g_z + \varepsilon^2\pi_{20}g_z + ..., \quad (3.24)$$
$$M_0(t) = g_z(x_0, x_0(t-h), z_0),$$
$$\pi_{00}g_z = f_y(x_0(0), x_0(-h), z_0(0) + \Pi_{00}(\tau_0)) - g_z(x_0(0), x_0(-h), z_0(0)),$$
....

After substituting (3.19)-(3.24) into (3.1), (3.2) and collecting the terms of the same power of ε, it remains to equate the same order terms to each other. We obtain six equations

$$\frac{dx_1}{dt} = A_0x_1 + B_0x_1(t-h) + L_0z_1 + A_0\bar{\pi}_{10}, \quad (3.25)$$

$$\frac{dz_0}{dt} = C_0x_1 + D_0x_1(t-h) + M_0z_1 + C_0\bar{\pi}_{10}, \quad (3.26)$$

$$\frac{d\Pi_{10}}{d\tau_0} = M_0(0)\Pi_{10} + \Pi_{10}^*(\tau_0), \quad (3.27)$$

$$\frac{d\Pi_{11}}{d\tau_1} = M_0(h)\Pi_{11} + \Pi_{11}^*(\tau_1), \quad (3.28)$$

$$\frac{d\pi_{20}}{d\tau_0} = \pi_{20}^*(\tau_0), \quad (3.29)$$

$$\frac{d\pi_{21}}{d\tau_1} = \pi_{21}^*(\tau_1), \quad (3.30)$$

where $\Pi_{10}^*, \Pi_{11}^*, \pi_{20}^*, \pi_{21}^*$ are still known boundary functions.

Assumption 1 (the matrix A is of Gurvitz type) implies that M_0^{-1} exists. Therefore equation (3.26) may be written in the form

$$z_1 = M_0^{-1}(\frac{dz_0}{dt} - C_0x_1 - D_0x_1(t-h) - C_0\bar{\pi}_{10}). \quad (3.31)$$

Using (3.31), we write (3.25) as follows

$$\frac{dx_1}{dt} = A^*x_1 + B^*x_1(t-h) + f_1^*(t), \quad (3.32)$$

where

$$A^* = A_0 - L_0M_0^{-1}C_0, \quad (3.33)$$
$$B^* = B_0 - L_0M_0^{-1}D_0, \quad (3.34)$$
$$f_1^* = A_0\bar{\pi}_{10} - L_0M_0^{-1}C_0\bar{\pi}_{10} + M_0^{-1}\frac{dz_0}{dt}. \quad (3.35)$$

Recalling

$$x_1(t) + \bar{\pi}_{10}(t) = 0, \quad t \in [-h, 0],$$

we get for x_1 the initial problem. Hence, x_1 may be determined in $[-h, T]$ uniquely. Note the $\dot{x}_1$ can have a jump at the point $t = h$, since f_1^* can have jump at this point. Moreover, the function z_1 (see (3.31)) can have jumps at $t = h$. Further, the general solution of equation (3.28) may be written as

$$\Pi_{11}(\tau_1) = e^{M_0(h)\tau_1}\nu_1 + \int_0^{\tau_1} e^{M_0(h)(\tau_1 - s)}\Pi_{11}^*(s)ds, \quad \tau_1 > 0,$$

where ν_1 is an arbitrary vector from R^m. It is easy to see Π_{11} is a boundary function in $\tau_1 > 0$. Setting $\Pi_{11}(\tau_1) = 0$ for $\tau_1 \le 0$, we define ν_1 as follows

$$\nu_1 = z_1(h+0) - z_1(h-0).$$

In other words, Π_{11} gives a possibility to stitch up a jump of function z_1 in the point $t = h$, so that

$$z_1 + \Pi_{11}(\tau_1)$$

is continuous function at the point $t = h$. Putting

$$\Pi_{10}(0) = -z_1(0), \qquad (3.36)$$

we find Π_{10} from (3.27), (3.36) uniquely by

$$\Pi_{10}(\tau_1) = -e^{M_0(0)\tau_0} z_1(0) + \int_0^{\tau_0} e^{M_0(0)(\tau_0 - s)} \Pi_{10}^*(s)ds, \ \ \tau_0 > 0,$$

To end, it is necessary to put

$$\pi_{20}(\tau_0) = -\int_{\tau_0}^{\infty} \pi_{20}^*(s)ds, \ \ \tau_0 > 0,$$

$$\pi_{20}(\tau_0) = -\int_0^{\infty} \pi_{20}^*(s)ds, \ \ \tau_0 \le 0,$$

$$\pi_{21}(\tau_1) = -\int_{\tau_1}^{\infty} \pi_{21}^*(s)ds, \ \ \tau_1 > 0,$$

$$\pi_{21}(\tau_1) = -\int_0^{\infty} \pi_{21}^*(s)ds, \ \ \tau_1 \le 0.$$

Now suppose that for some integer k the functions

$$x_0, x_1, \dots, x_k,$$

$$z_0, z_1, \dots, z_k,$$

$$\pi_{r0}, \pi_{r1}, \dots, \pi_{r,r-1}, \ \ r = 1, \dots, k,$$

$$\Pi_{r0}, \Pi_{r1}, \dots, \Pi_{rr}, \ \ r = 0,1, \dots, k,$$

$$\pi_{k+1,0}, \pi_{k+1,1}, \dots, \pi_{k+1,k}$$

are defined. To carry out the following step let us consider (3.1), (3.2) and separate the terms of power order ε^{k+1}. Omitting the all boundary functions in (3.1), (3.2), we get for x_{k+1}, z_{k+1} the system

$$\frac{dx_{k+1}}{dt} = A_0 x_{k+1} + B_0 x_{k+1}(t-h) + L_0 z_{k+1} + f_{k+1}, \qquad (3.37)$$

$$\frac{dz_k}{dt} = C_0 x_{k+1} + D_0 x_{k+1}(t-h) + M_0 z_{k+1} + g_{k+1}(t), \qquad (3.38)$$

where $f_{k+1}, g_{k+1}, dz_k / dt$ are already defined as functions of $\bar{\pi}_{sl}(s = 0,1,\dots,k+1; l < s)$ and hence these functions can have jumps at the points $t = h, 2h, \dots, kh$. From (3.38) we have

$$z_{k+1} = M_0^{-1}\left(\frac{dz}{dt} - C_0 x_{k+1} - D_0 x_{k+1}(t-h) - g_{k+1}(t)\right). \qquad (3.39)$$

Substituting z_{k+1} into (3.37) we obtain

$$\frac{dx_{k+1}}{dt} = A^* x_{k+1}(t-h) + L_0 z_{k+1} + f_{k+1}^*(t), \qquad (3.40)$$

$$x_{k+1}(t) + \bar{\pi}_{k+1,0} + \bar{\pi}_{k+1,1} + \dots + \bar{\pi}_{k+1,k} = 0, \ \ t \in [-h, 0], \qquad (3.41)$$

where $f^*(t)$ can have jumps at the points $t = h, 2h, \dots, kh$. The function x_{k+1} is defined uniquely from (3.40), (3.41), and then z_{k+1} will be found from (3.39). Obviously, z_{k+1} can have jumps in the points $t = h, 2h, \dots, (k+1)h$. Next, selecting in (3.2) the main boundary functions at power order ε^{k+1}, we get easy

$$\frac{d\Pi_{k+1,s}}{d\tau_s} = M_0(sh)\Pi_{k+1,s} + \Pi_{k+1,s}^*(\tau_s), \ \ \tau_s > 0, \qquad (3.42)$$

$$s = 0,1,\dots,k+1,$$

where $\Pi_{k+1,s}^*$ are the known boundary functions. General solution of the system (3.42) can be presented in the form

$$\Pi_{k+1,s}(\tau_s) = e^{M_0(sh)\tau_s} \nu_s + \int_0^{\tau_s} e^{M_0(sh)(\tau_s - \mu)} \Pi_{k+1,s}^*(\mu) d\mu, \ \ \tau_s > 0,$$

$$s = 0,1,\dots,k+1,$$

For $s = 1,2,\dots,k+1$ constants $\nu_s \in R^m$ should be determined as follows

$$\nu_s = z_{k+1}(sh+0) - z_{k+1}(sh-0)$$

to stitch up the jumps of function z_{k+1} at the points $t = h, 2h, \dots, (k+1)h$. Further, set $\nu_0 = -z_{k+1}(0)$. Finally, we should collect in the right-hand part of (3.1) all boundary functions of the power order ε^{k+2}

$$\pi_{k+2,0}^*, \pi_{k+2,1}^*, \dots, \pi_{k+2,k+1}^*$$

and put

$$\pi_{k+2,s} = -\int_{\tau_s}^{\infty} \pi^*_{k+2,s}(\mu)\,d\mu, \quad \tau_s > 0,$$

$$\pi_{k+2,s} = -\int_{\tau_s}^{\infty} \pi^*_{k+2,s}(\mu)\,d\mu, \quad \tau_s > 0.$$

The present approach allows to determine an approximate solution x_*, z_* in the form (2.5), (2.6). Moreover,

$$\frac{dx_*}{dt} = f(x_*, x_*(t-h), z_*) + r(t,\varepsilon), \tag{3.43}$$

$$\varepsilon\frac{dz_*}{dt} = g(x_*, x_*(t-h), z_*) + R(t,\varepsilon), \tag{3.44}$$

and for some $C > 0$, $\varepsilon_0 > 0$ the estimates

$$\max_{0\le t\le T} \|r(t,\varepsilon)\| \varepsilon^{-1-N} \le C, \tag{3.45}$$

$$\max_{0\le t\le T} \|R(t,\varepsilon)\| \varepsilon^{-1-N} \le C, \tag{3.46}$$

where C is independent of $\varepsilon \in (0,\varepsilon_0]$, can be easy proved.

4. Estimate of Remainder Terms

To estimate $x - x_*, z - z_*$ let us introduce the new variables

$$u = x - x_*, \upsilon = z - z_*. \tag{4.1}$$

Then for u, v we get a new Cauchy problem

$$\frac{du}{dt} = f(x_* + u, x_*(t-h) + u(t-h), z_* + \upsilon) + r(t,\varepsilon), \tag{4.2}$$

$$\varepsilon\frac{d\upsilon}{dt} = g(x_* + u, x_*(t-h) + u(t-h), z_* + \upsilon) + R(t,\varepsilon), \tag{4.3}$$

$$u(t) = 0 \ \ (-h \le t \le 0), \ \ \upsilon(0) = 0. \tag{4.4}$$

Our purpose is to prove existence of such a constants $C > 0$, $\varepsilon_0 > 0$, that for all $\varepsilon \in (0,\varepsilon_0]$ an estimate

$$\|u(t,\varepsilon)\|_{C[-h,T]} + \|\upsilon(t,\varepsilon)\|_{C[0,T]} \le C\varepsilon^{N+1} \tag{4.5}$$

holds. By linearization the system (4.2), (4.3) at the point

$$x = u(t) \in C[-h,T], \quad z = z_0(t) + \Pi_{00}(\tau_0) \in C[0,T]$$

we obtain a new nonlinear system

$$\frac{du}{dt} = A(t,\tau_0)u + B(t,\tau_0)u(t-h) + L(t,\tau_0)\upsilon + $$
$$+ \mathcal{G}(u, U(t-h), \upsilon, t, \tau_0, \varepsilon), \tag{4.6}$$

$$\varepsilon\frac{d\upsilon}{dt} = C(t,\tau_0)u + D(t,\tau_0)u(t-h) + M(t,\tau_0)\upsilon +$$
$$+ \mathcal{H}(u, u(t-h), \upsilon, t, \tau_0, \varepsilon), \tag{4.7}$$

$$\mathcal{G} = f(x_* + u, x_*(t-h) + u(t-h), \upsilon + z_*) - \frac{dx_*}{dt} -$$
$$- f_x(u_0, u_0(t-h), z_0 + \Pi_{00})u - f_y(u_0, u_0(t-h), z_0 + \Pi_{00})u(t-h) -$$
$$- f_z(u_0, u_0(t-h), z_0 + \Pi_{00})\upsilon + (\ldots), \tag{4.8}$$

$$\mathcal{H} = g(x_* + u, x_*(t-h) + u(t-h), \upsilon + z_*) - \varepsilon\frac{dz_*}{dt} -$$
$$- g_x(u_0, u_0(t-h), z_0 + \Pi_{00})u - g_y(u_0, u_0(t-h), z_0 + \Pi_{00})u(t-h) -$$
$$- g_z(u_0, u_0(t-h), z_0 + \Pi_{00})\upsilon - (\ldots), \tag{4.9}$$

where by (...) we denote the terms of the second order with respect to u, v.

It is easy to see the nonlinear functions $\mathcal{G}$, $\mathcal{H}$ possess the two principal properties.

i) There exist such $C > 0$, $\varepsilon_0 > 0$ that for all $\varepsilon \in (0,\varepsilon_0]$

$$\|\mathcal{G}(0,0,0,t,\tau_0,\varepsilon)\|_{C[0,T]} + \|\mathcal{H}(0,0,0,t,\tau_0)\|_{C[0,T]} \le C\varepsilon^{N+1};$$

ii) for any $\Delta > 0$ there exist such $\delta > 0$, $\varepsilon_0 > 0$ that for all

$$\|u_1\|, \|u_2\|, \|\upsilon_1\|, \|\upsilon_2\| \le \delta, \ \ 0 < \varepsilon \le \varepsilon_0$$

the inequalities

$$\|\mathcal{G}(u_1, u_1(t-h), \upsilon_1, t, \tau_0, \varepsilon) - \mathcal{G}(u_2, u_2(t-h), \upsilon_2, t, \tau_0, \varepsilon)\|_{C[0,T]} \le$$
$$\le \Delta(\|u_1 - u_2\|_{C[-h,T]} + \|\upsilon_1 - \upsilon_2\|_{C[0,T]}), \tag{4.10}$$

$$\|\mathcal{H}(u_1, u_1(t-h), \upsilon_1, t, \tau_0, \varepsilon) - \mathcal{H}(u_2, u_2(t-h), \upsilon_2, t, \tau_0, \varepsilon)\|_{C[0,T]} \le$$

$$\le \Delta(\|u_1 - u_2\|_{C[-h,T]} + \|\upsilon_1 - \upsilon_2\|_{C[0,T]}), \quad (4.11)$$

are valid.

Let $\mathcal{K}(t,s,\varepsilon)$ be Cauchy matrix for time-delay system

$$\frac{du}{dt} = A(t,\tau_0)u + B(t,\tau_0)u(t-h), \quad s < t \le T,$$

with

$$\mathcal{K}(t,s,\varepsilon) = \begin{cases} 0, & t < s, \\ I_n, & t = s. \end{cases}$$

Denote by G_0 the triangle

$$G_0 = \{(t,s) | 0 \le s \le t \le T\},$$

and let

$$\mathcal{K}_0(t,s,\varepsilon) = \begin{cases} \mathcal{K}(t,s,\varepsilon), & (t,s) \in G_0, \\ 0, & (t,s) \in G_0. \end{cases}$$

Finally, let $U(t,s,\varepsilon)$ $(0 \le s \le t \le T)$ be a fundamental matrix for singularly perturbed system

$$\varepsilon \frac{dU}{dt} = M(t,\tau_0), \quad s < t \le T,$$

As it is shown [2], the estimate

$$\|U(t,s,\varepsilon)\| \le C exp(-\mu(t-s)/\varepsilon) \quad (C>0,\ \mu>0,\ 0 \le s \le t \le T) \quad (4.12)$$

is true, where C, μ are independent of $(t,s) \in G_0$, $\varepsilon \in (0,\varepsilon_0]$.

If

$$C_0[-h,T] = \{u \in C([-h,T],R^n) \ \ with \ \ u = 0, -h \le t \le 0\},$$

$$C_0[0,T] = \{\upsilon \in C([0,T],R^m) \ \ with \ \ \upsilon(0) = 0\},$$

then the Cauchy problem (4.6), (4.7), (4.4) is equivalent to the integral system of the equations

$$u = \int_0^t \mathcal{K}_0(t,s,\varepsilon)[L(p,p/\varepsilon)\upsilon(p) +$$

$$+ G(u(p),u(p-h),p,p/\varepsilon)]dp \quad (-h \le t \le T) \quad (4.13)$$

$$\upsilon = \frac{1}{\varepsilon}\int_0^t U(t,s,\varepsilon)[C(s,s/\varepsilon)u(s) + D(s,s/\varepsilon)u(s-h) +$$

$$+ H(u(s),u(s-h),\upsilon(s),s,s/\varepsilon)]ds \quad (4.14)$$

$$(0 \le t \le T), \quad u \in C_0[-h,T], \quad \upsilon \in C_0[0,T].$$

To analyze a problem (4.13), (4.14), first transform equation (4.14). Taking into account we have

$$u(s-h) = \int_0^s \mathcal{K}_0(s-h,p,\varepsilon)\{L(p,p/\varepsilon)\upsilon(p) +$$

$$+ \mathcal{G}(u(p),u(p-h),p,p/\varepsilon),\varepsilon]dp \quad s \in [0,T]. \quad (4.15)$$

Therefore, substituting (4.13), (4.15) into (4.14) we obtain

$$\upsilon = \frac{1}{\varepsilon}\int_0^t\int_0^s U(t,s,\varepsilon)\{C(s,s/\varepsilon) + \mathcal{K}_0(s,p,\varepsilon)L(p,p/\varepsilon) +$$

$$+ D(s,s/\varepsilon)\mathcal{K}_0(s-h,p,\varepsilon)L(p,p/\varepsilon)\}\upsilon(p)dp\,ds +$$

$$+ \frac{1}{\varepsilon}\int_0^t\int_0^s U(t,s,\varepsilon)\mathcal{K}_0(s,p,\varepsilon)C(s,s/\varepsilon) \times$$

$$\times G(u(p),u(p-h),\upsilon(p),p,p/\varepsilon)dp\,ds +$$

$$+ \frac{1}{\varepsilon}\int_0^t U(t,s,\varepsilon) + \mathcal{H}(u(s),u(s-h),\upsilon(s),s,s/\varepsilon)ds.$$

Changing the integration order in the first integral, we have

$$\upsilon = \frac{1}{\varepsilon}\int_0^t\int_p^t U(t,s,\varepsilon)\{C(s,s/\varepsilon) + \mathcal{K}_0(s,p,\varepsilon)L(p,p/\varepsilon) +$$

$$+ D(s,s/\varepsilon)\mathcal{K}_0(s-h,p,\varepsilon)L(p,p/\varepsilon)\}ds\,\upsilon(p)dp + Q(u,\upsilon,\varepsilon), \quad (4.16)$$

where Q are nonlinear terms. It is easy to verify that due to estimate (4.12), the kernel

$$\mathcal{K}^*(t,p,\varepsilon) = \frac{1}{\varepsilon}\int_p^t U(t,s,\varepsilon)\{C(s,s/\varepsilon) + \mathcal{K}_0(s,p,\varepsilon)L(p,p/\varepsilon)\}ds,$$

is a sufficiently small number.

Hence, equation (4.16) can be rewritten in the form

$$\upsilon = \int_0^t \mathcal{K}^*(t,p,\varepsilon)\upsilon(p)dp + Q(u,\upsilon,\varepsilon), \quad (4.17)$$

and for u from (4.13) we have

$$u = \int_0^t \mathcal{K}^{**}(t,p,\varepsilon)\upsilon(p)\,dp + P(u,\upsilon,\varepsilon),$$
(4.18)

where

$$\mathcal{K}^{**}(t,p,\varepsilon) = \mathcal{K}_0(s,p,\varepsilon)L(p,p/\varepsilon).$$

It is extremely important to emphasize that Q and P are subjected the properties i), ii).

At last, introduce two nonlinear operators

$$S(u,\upsilon,\varepsilon) = \int_0^t \mathcal{K}^{*}(t,p,\varepsilon)\upsilon(p)\,dp + Q(u,\upsilon,\varepsilon),$$

$$W(u,\upsilon,\varepsilon) = \int_0^t \mathcal{K}^{**}(t,p,\varepsilon)\upsilon(p)\,dp + P(u,\upsilon,\varepsilon),$$

$$u \in C_0[-h,T], \quad \upsilon \in C_0[0,T].$$

Obviously,

$$S(0,0,\varepsilon) = O(\varepsilon^{N+1}), \ W(0,0,\varepsilon) = (\varepsilon^{N+1}). \tag{4.19}$$

Therefore the successive approximations

$$u_1 = S(0,0,\varepsilon), \ \upsilon_1 = W(0,0,\varepsilon),$$
$$u_2 = S(u_1,\upsilon_1,\varepsilon), \ \upsilon_2 = W(u_1,\upsilon_1,\varepsilon),$$
$$u_3 = S(u_2,\upsilon_2,\varepsilon), \ \upsilon_3 = W(u_2,\upsilon_2,\varepsilon),$$
....

converge to some $u^* \in C_0[-h,T]$, $\upsilon^* \in C_0[0,T]$ and, in addition,

$$\|u^*\|_{C[-h,T]} \le C^*\varepsilon^{N+1}, \quad \|\upsilon^*\| \le C^*\varepsilon^{N+1},$$

where C^* is independent of $\varepsilon \in (0,\varepsilon_0]$.

Thus, we prove the following assertion.

Theorem 4.1. *Under assumptions 1-4 there exist such positive numbers $\varepsilon_0 > 0$, $C_0 > 0$ that for all $\varepsilon \in (0,\varepsilon_0]$ Cauchy problem (2.1)-(2.4) has a unique solution*

$$x = x_* + u^*(t,\varepsilon), \quad z = z_* + \upsilon^*(t,\varepsilon),$$

moreover, estimates (4.5) are true.

Remark 5.1. The estimates (4.5) will be valid, if in the expansion (2.5) to omitt the sum

$$\pi_{N+1,0}(\tau_0) + \pi_{N+1,1}(\tau_1) + \ldots + \pi_{N+1,N}(\tau_N).$$

5. Conclusions

A singular perturbations theory extends the sphere of influence the last four decades. A number of science fields as biology, chemical kinetics, control theory widely use this theory. Due to new results on boundary functions for partial equations the singular theory has spread among the great number of the the engineering problems. But one domain remains still unnoticed. It is the singular perturbed delay- time systems which plays important role in the control theory. We try to fill this gap. So, presented paper deals with singularly perturbed systems with one time-delay h. The asymptotical expansions of solutions of Cauchy problem of order N uniformly bounded with respect to $t \in [-h,T]$ and $\varepsilon \in (0,\varepsilon_0]$ are found. It turns out these expansions contain the new additional boundary functions in the neighborhoods of $t = h$, $2h$, ... , Nh. The approach presented is the generalization of the well known method proposed by A.B. Vasileva [2,3] and coinsides with it for ordinary differential equations. The submitted method can be used for a number of problems of control theory, combustion in rocket motors [9], chemical reactors [11] et al.

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